Algebra Qualifying Exam: January 28, 2021

Instruction: There are six problems, each is worth equal value. Please show all necessary details for your work. You are allowed to freely quote well-known results from 790 and 791 (say Sylow Theorems or the classification of finite abelian groups) but should refrain from using more advanced results without justification.

Question 1. a) Write down all possible Jordan canonical forms for a 5 by 5 matrix A such that $A^3 = 0$ (the blocks should have non-increasing size down the diagonal. We work over complex numbers).
b) Let N be a nilpotent matrix over any field. Prove that $I + N$ is diagonalizable if and only if $N = 0$ (I is the identity matrix of the same size).

Question 2. A (complex) matrix is called *fake Hermitian* if: 1) all of its eigenvalues are real and 2) any two eigenvectors corresponding to different eigenvalues are orthogonal.
a) Prove that Hermitian matrices (namely, $A = \overline{A}^T$) are fake Hermitian.
b) Can you find a fake Hermitian matrix that is not similar to any Hermitian matrix?

Question 3. a) Let A be a real n by n matrix such that $A^2 = 3A$. Prove that A is similar (over the real numbers) to a diagonal matrix whose diagonal entries are 3 or 0.
b) Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ (note that $A^2 = 3A$). Find a real matrix S so that $S^{-1}AS$ has the form as in part a).

Question 4. Let k be a field and $R = k[X,Y]/I$ where I is the ideal (X^2, Y^2). Let x, y be the image of X, Y in R.
a) Prove that R has only one maximal ideal $\mathfrak{m} = (x, y)$.
b) Prove that any ideal of R strictly contained in $\mathfrak{m}$ is principal.

Question 5. Find the Galois group of $x^3 + x + 2021$ over $\mathbb{Q}$. $(2021 = 43 \times 47)$

Question 6. For a field F let $GL_n(F)$ be the group of invertible matrices with entries in F of size n under matrix multiplication.
a) Show that any finite subgroup of $GL_1(F)$ is cyclic.
b) Show that $GL_n(F)$ always contains a finite subgroup H that are minimally generated by n generators (so H can be generated by n generators and no $n-1$ elements generate H).

ALGEBRA QUALIFYING EXAM: JANUARY 14, 2020

Each problem is of equal value. You must show all work to receive full credit.

1. (a) Describe all possible Jordan canonical forms for 4×4 matrices over $\mathbb{C}$ whose minimal polynomial has only two (possibly repeated) irreducible factors. Give an example in each case.
   (b) Describe the precise circumstances under which the Jordan form of a linear operator equals its rational canonical form.

2. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator.
   (a) Suppose the matrix of T with respect to the standard basis of $\mathbb{R}^3$ equals $\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & -\gamma \\ 0 & \gamma & \beta \end{pmatrix}$, with $\alpha, \beta, \gamma \in \mathbb{R}$. Prove that T is a normal operator.
   (b) Suppose the matrix of T with respect to the standard basis is $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$. Prove that T is a normal operator and find an orthonormal basis B for $\mathbb{R}^3$ such that the matrix of T with respect to B has the form in part (a).

3. Let A be an $n \times m$ non-zero matrix with coefficients in the field F. Prove that the rank of A equals the largest positive integer t such that there is a $t \times t$ submatrix of A with non-zero determinant.

4. Given a field K, consider the group $GL_n(K)$ of invertible $n \times n$ matrices with entries in K.
   (a) Show that the center of $GL_n(K)$ is $\{\alpha I_n \mid \alpha \in K^\times\}$, where I_n is the $n \times n$ identity matrix.
   Hint: Consider the matrix $I_n + A_{ij}$, where A_{ij} is the matrix whose (i, j)-th entry is 1, and all other entries are 0.
   (b) Show that $|SL_2(\mathbb{F}_3)| = 24$, where $\mathbb{F}_3$ is the field with three elements, and $SL_2(\mathbb{F}_3)$ is the subgroup of $GL_2(\mathbb{F}_3)$ of 2×2 matrices with determinant 1.
   (c) Use (a) to prove that $SL_2(\mathbb{F}_3)$ is not isomorphic to the symmetric group S_4.

5. Given a commutative ring R with 1, the *Jacobson radical*, $J(R)$, is the intersection of all maximal ideals of R.
   (a) Find the Jacobson radical of $\mathbb{Z}/12\mathbb{Z}$, and of $\mathbb{Q}[x, y]/\langle x^2, y - 5\rangle$, with full justification.
   (b) Fix an element r of a commutative ring R with 1. Prove that $r \in J(R)$ if and only if for every $s \in R$, the element $rs - 1$ is a unit of R.

6. Fix a field K, whose characteristic is not 2. Consider $\alpha, \beta \in K$, neither of which is a square in K. Prove that $K(\sqrt{\alpha}, \sqrt{\beta})$ has degree 2 over K if and only if $\alpha\beta$ is a square in K.

FALL 2020 ALGEBRA QUALIFYING EXAMINATION

Each problem is of equal value. You must show all work to receive full credit.

1. Let $G := \{I := C_1, C_2, \ldots, , C_r\}$ be a finite set of invertible $n \times n$ complex matrices, such that G is a finite group under matrix multiplication. Here I denotes the $n \times n$ identity matrix.
   (i) Prove that each C_i is diagonalizable.
   (ii) Show that trace $(C_i^{-1}) = \overline{\text{trace}(C_i)}$, for all i, where $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$.
   (iii) Let $\langle -, - \rangle$ denote the standard inner product on $\mathbb{C}^n$. Define $[v, w] := \frac{1}{|G|} \sum_{i=1}^{r} \langle C_i v, C_i w \rangle$, for all $v, w \in \mathbb{C}^n$. Prove that $[-, -,]$ is an inner product on $\mathbb{C}^n$ and that $[v, w] = [C_i v, C_i w]$, for all $v, w \in \mathbb{C}^n$ and $C_i \in G$.

2. Let T be a linear operator on a finite dimensional complex inner product space V. Recall that T is a *normal* operator if $TT^* = T^*T$, where T^* denote the adjoint of T. Prove that T is a normal operator if and only if $||T(v)|| = ||T^*(v)||$, for all $v \in V$. (Hint: use the fact that $T^*T - TT^*$ is self-adjoint.). Give an example of a normal operator on a three dimensional complex space that is not self adjoint.

3. For the real valued matrix $A = \begin{pmatrix} 0 & 3 & 0 & 15 \\ -1 & 0 & 5 & 5 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & -1 & -1 \end{pmatrix}$, find an invertible 4×4 real matrix P such that $P^{-1}AP$ is in rational canonical form with respect to the elementary divisors of A. How does this form differ, if at all , from the invariant factor rational canonical form of A. Justify your answer.

4. Let G be a group order $3n$ where n is an **odd** number.
   (a) Prove that any subgroup of order n in G is normal.
   (b) Give an example, with full justification, to show that the statement in part a) is no longer true if n is even.

5. Let R be a PID. Let I be a nonzero proper ideal in R and $S = R/I$. Prove that the following are equivalent:
   (a) S is *indecomposable*. (Namely, S cannot be written as direct sum of two non-zero ideals $S = I \oplus J$.)
   (b) $I = (f^n)$ where f is a prime element and n is some positive integer.

6. Construct a field F with nine elements, with full justification. Then prove that the map $f : F \to F$ defined by $f(\alpha) = \alpha^3$, for all $\alpha \in F$, is an automorphism (i.e., an isomorphism from F to itself).

ALGEBRA QUALIFYING EXAM: JANUARY 17, 2019

Each problem is of equal value. You must show all work to receive full credit.

1. Prove that any group of order 15 must be cyclic. You may freely use Sylow's Theorems, as well as basic properties of orders of elements.

2. Let R denote the ring $\mathbb{C}[t, t^{-1}]$ of Laurent polynomials over $\mathbb{C}$.
   (i) Give a rigorous proof that R is isomorphic to $\mathbb{C}[x, y]/(xy - 1)$.
   (ii) Show that every ideal in R is a principal ideal.
   (iii) Give an example (with proof) of a non-zero prime ideal in R.

3. Consider a field K, and an irreducible polynomial $f(x) \in K[x]$. Prove that if L is a field extension of K such that $[L : K]$ is relatively prime to the degree of $f(x)$, then $f(x)$ remains irreducible in $L[x]$.

4. Let V denote the vector space of 2×2 matrices over $\mathbb{R}$ equipped with the inner product defined by $\langle A, B\rangle := \mathrm{trace}(A^t B)$, where A^t denotes the transpose of A. Let $T : V \to V$ be the linear operator on V that takes $A \in V$ to the matrix obtained from A by interchanging the rows of A. Determine (with full justification) whether or not T is orthogonally diagonalizable. If it is, find an orthonormal basis B for V such that the matrix of T with respect to that basis is diagonal.

5. Let V be a vector space over an infinite field and $v_1, \ldots, v_n \in V$ be finitely many non-zero vectors. Prove that there exists $f \in V^*$ such that $f(v_i) \neq 0$, for all $1 \leq i \leq n$. Here V^* denotes the dual space of V.

6. Let V be a finite dimensional vector space over the field F and suppose $T : V \to V$ is a linear operator on V.
   (i) Prove that if the minimal polynomial of T is $x^2 - x$, then the kernel of T has a T-invariant complement.
   (ii) Give an example where V has a proper T-invariant subspace W such that W does not have a T-invariant complement.
   (iii) If $V = \mathbb{R}^4$, give an example where T has only finitely many T-invariant subspaces. Hint: It may be useful to consider the case that T is a normal operator on $\mathbb{R}^4$.

ALGEBRA QUALIFYING EXAMINATION: AUGUST 20, 2019

Each problem is of equal value. You must show all work to receive full credit.

1. Let $A = \begin{pmatrix} 4 & -3 & -3 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{pmatrix}$. Prove that the set of 3×3 matrices over $\mathbb{R}$ that commute with A form a subspace of $\mathrm{M}_n(\mathbb{R})$ and calculate (with proof) the dimension of that subspace.

2. Let V be a finite dimensional vector space over the complex numbers and T a linear operator on V. Suppose that the subspace of V generated by all of the eigenvectors of T is one-dimensional. Prove that the minimal polynomial of T is $(x - \lambda)^n$, for some complex number λ and n, the dimension of V.

3. Let V denote a finite dimensional inner product space over $\mathbb{R}$.
   (i) For a subspace $W \subseteq V$, prove that $V = W \oplus W^{\perp}$.
   (ii) Recall that for $v \in V$, if we write $v = w + w'$, with $w \in W$ and $w' \in W^{\perp}$, w is the *orthognoal projection* of v onto W. Now, suppose that V is the space of real 2×2 matrices with inner product $\langle A, B \rangle = \mathrm{trace}(A^t B)$ and W is the subspace of symmetric 2×2 matrices. Find the orthogonal projection of $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ onto W.

4. Recall that if G is a group and $x, y \in G$, x and y are conjugate if $y = gxg^{-1}$ for some $g \in G$, and that conjugacy is an equivalence relation on G. Let $[x]$ denote the conjugacy class of an element $x \in G$; i.e., $[x] = \{gxg^{-1} \mid g \in G\}$.
   (i) Prove that for $x \in G$, $[x]$ has exactly one element if and only if $x \in Z(G)$, where $Z(G)$ denotes the center of G.
   (ii) More generally, prove that the number of elements in $[x]$ equals $|G : C_G(x)|$, where $C_G(x)$ is the *centralizer of x in G*, i.e., $C_G(x) = \{g \in G \mid gxg^{-1} = x\}$. *Hint*: When is gxg^{-1} equal to hxh^{-1}?
   (iii) Suppose that G is a finite group of odd order, and suppose that N is a normal subgroup of G of order 3. Prove that $N \leq Z(G)$. *Hint*: What does the normality of N say about the conjugacy classes of elements $x \in N$? How can one use this to partition N?

5. Consider the ring $R = \mathbb{Q}[x, y, z]/I$, where $I = (x^2 y - z^5)$.
   (i) Prove that R is not a field. Then decide whether R is an integral domain, and justify your answer.
   (ii) Given $f \in \mathbb{Q}[x, y, z]$, let $\overline{f}$ denote the element $f + I$ of R. Is the ideal $J = (\overline{x}, \overline{y})$ a prime ideal of R? Prove your answer.
   (iii) Show that $\overline{z^5} \in J^2$, but that $\overline{z^4} \notin J^2$.

6. Consider the polynomial $p(x) = x^3 - 2 \in \mathbb{Q}[x]$. Find $\alpha, \beta \in \mathbb{C}$ for which $F = \mathbb{Q}(\alpha, \beta)$ is a splitting field for $p(x)$, and prove that this is the case. Then find, with justification, the degree of the field extension $\mathbb{Q} \subseteq F$.

Algebra Qualifying Exam: January 9, 2018

Each problem is of equal value. You must show all work to receive full credit.

Question 1. Let $R = \mathbb{C}[x, y, z]$ be the ring of polynomials in 3 variables over the complex numbers.
a) Show that $I = (x, y)$ is a prime ideal in R.
b) Let $J = (x^2, y^2)$. Prove that for any collection of polynomials $\{f_1, f_2, \ldots, f_n\}$ such that the product $f_1 f_2 \ldots f_n$ is in J, we can find a subset of at most 3 polynomials whose product is already in J.
c) Let $K = (x^2y^2, y^2z^2, z^2x^2)$. Prove that for any collection of polynomials $\{f_1, f_2, \ldots, f_n\}$ such that the product $f_1 f_2 \ldots f_n$ is in K, we can find a subset of at most 9 polynomials whose product is already in K.

Question 2. Consider $G := \mathbb{Z} \times \mathbb{Z}$ regarded as an abelian group.
a) Find an element $(a, b) \neq (0, 0)$ such that the factor group $G/\langle(a, b)\rangle$ is torsion free, i.e., there are no elements of finite order.
b) Suppose $a, b \in \mathbb{Z}$ are nonzero. Set $H_1 := \langle(a, 0)\rangle$ and $H_2 := \langle(0, b)\rangle$. Prove that $G/(H_1 \times H_2)$ is isomorphic to $\mathbb{Z}/\langle GCD(a, b)\rangle \times \mathbb{Z}/\langle LCM(a, b)\rangle$.

Question 3. Let the complex number ϵ be a primitive 5^{th} root of unity, i.e., $\epsilon^5 = 1$, but $\epsilon^j \neq 1$, for $1 \leq j \leq 4$. Set $F := \mathbb{Q}(\epsilon)$.
a) Find $[F : \mathbb{Q}]$.
b) Determine (with proof) whether or not there exists a field K such that $\mathbb{Q} \subsetneq K \subsetneq F$.

Question 4. 1. Let V be a finite dimensional vector space over a field F and $T : V \to V$ be a linear transformation. Let $F[T]$ denote the ring of all linear transformations on V that can be expressed as a polynomial in T. Assume that no nonzero subspace of V is mapped into itself by T.
a) If $0 \neq S \in F[T]$, show that the null space of S is zero.
b) Prove that $F[T]$ is a field.
c) Show that $[F[T] : F]$, the degree of $F[T]$ over F, equals $\dim_F(V)$.

Question 5. Let V be the vector space of all polynomials of degree at most 3 over the complex numbers and $T : V \to V$ be the linear transformation $T(f) = f + f''$. Describe, with proof, the Jordan Canonical Form or T.

Question 6. Let V be the space of n by n matrix with entries in $\mathbb{C}$. For a *subset* $S \subseteq V$, let $C(S) = \{B \in V |\ AB = BA\ \forall A \in S\}$. Let $A \in V$ be a matrix. Prove that each element in $C(C(\{A\}))$ is of the form $p(A)$ with some polynomial $p \in \mathbb{C}[t]$.

ALGEBRA QUALIFYING EXAM: AUGUST 16, 2018

Each problem is of equal value. You must show all work to receive full credit.

1. (a) How many distinct (up to isomorphism) finite abelian groups of order 14553000 are there? Hint: $14553000 = 2^3 \cdot 3^3 \cdot 7^2 \cdot 5^3 \cdot 11$.
(b) Let G be a finite abelian group and p be a prime number dividing $|G|$. Prove that G has a *unique* Sylow p-subgroup (without citing any parts of the Sylow theorems).

2. For each of the following, provide a proof if the statement is correct or provide a counter-example or justification, if the statement is false.

(a) Let A denote the ring of continuous real-valued functions on the open unit interval $(0, 1)$. For $0 < \alpha < 1$, let $I_\alpha = \{f \in A \mid f(\alpha) = 0\}$.
  (i) Is I_α a maximal ideal?
  (ii) Is $I_{1/2} \cap I_{\pi/4}$ a prime ideal?
  (iii) Is (0) a prime ideal ?

(b) Is the ideal $\langle 2, x^2 + x + 1\rangle$ in $\mathbb{Z}[x]$ a maximal ideal?

(c) Is the ideal generated by the class of x in the quotient ring $\mathbb{C}[x, y]/\langle xy\rangle$ prime?

3. Let $F = \mathbb{Z}/2\mathbb{Z}$ denote the field with two elements and set $f(x) = x^5 + x^2 + 1 \in F[x]$.

(a) Show that $f(x)$ is irreducible over F.

(b) Let α be a root of $f(x)$ in some larger field. Express the element $\alpha \cdot (\alpha^4 + \alpha + 1)^{-1}$ in $F(\alpha)$ as a polynomial in α over F of *minimal* degree.

4. Let $p > 0$ be a prime number and set $F := \mathbb{Z}/p\mathbb{Z}$, the field with p elements. Find the number of 2×2 matrices with entries in F such that $A^2 = I$.

5. Let V be the space of 3×3 matrices over the field F and W the space of 2×2 matrices over F. Let X denote the set of all linear transformations $T : V \to W$ such that $T(A) = T(A^t)$, for all $A \in V$. Here, A^t denotes the transpose of A. Show that X is vector space and find the dimension of X.

6. Consider the matrix $A := \begin{pmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{pmatrix}$ with entries in $\mathbb{C}$. Find a matrix P such that $P^{-1}AP$ is in Jordan canonical form. Describe how you would find all matrices Q such that $Q^{-1}AQ$ is in Jordan canonical form.

ALGEBRA QUALIFYING EXAM: JANUARY 13, 2017

All answers must be fully justified to receive full credit. All problems are of equal value.

1. Let G be a finite group with $|G| = p^n$, p prime.
   (i) Prove that $Z(G)$, the center of G, is non-trivial.
   (ii) Prove that if $N \subseteq G$ is a normal subgroup of order p, then $N \subseteq Z(G)$. Hint: Let G act on N.
   (iii) Give an example of a non-abelian group group of order p^n whose center contains more than one normal subgroup of order p.

2. Let R be an integral domain. Suppose there exists a nonzero, non-unit $a \in R$ such that for every $r \in R$, there exists a unit u and $n \geq 0$ such that $r = ua^n$. Such a ring is called a *discrete valuation ring*. Set $P := aR$. Let $R[\frac{1}{a}]$ denote the ring of polynomial expressions in $\frac{1}{a}$ with coefficients in R. Note, $\frac{1}{a} \notin R$.
   (i) Prove that P is a maximal ideal, and in fact, the only maximal ideal.
   (ii) Prove that $\bigcap_{n=1}^{\infty} P^n = (0)$.
   (iii) Prove that $R[\frac{1}{a}]$ is a field.
   (iv) Let R be the ring of formal power series over $\mathbb{Q}$. Thus, a typical element in R is of the form $\sum_{i=0}^{\infty} \alpha_i x^i$, with $\alpha_i \in \mathbb{Q}$ and with addition and multiplication given just like for polynomials. Prove that R is a discrete valuation ring.

3. Let $\mathbb{Z}^n$ denote the free abelian group of rank n, its elements being row vectors of length n. Let A be an $r \times n$ matrix over $\mathbb{Z}$ and write K_A for the subgroup of $\mathbb{Z}^n$ generated by the rows of A.
   (i) Suppose $B := PAQ$, where P is an $r \times r$ invertible matrix over $\mathbb{Z}$ and Q is an invertible $n \times n$ matrix over $\mathbb{Z}$. Prove that $\mathbb{Z}^n/K_A$ and $\mathbb{Z}^n/K_B$ are isomorphic as abelian groups.
   (ii) Suppose $A = \begin{pmatrix} 4 & -2 & 4 \\ 2 & 4 & 4 \end{pmatrix}$. Write $\mathbb{Z}^3/K_A$ as a direct sum of cyclic groups.

4. Let $f(x) := x^3 - 9x + 3 \in \mathbb{Q}[x]$ and α a root of $f(x)$.
   (i) Prove that $f(x)$ is irreducible over $\mathbb{Q}$.
   (ii) In the field $\mathbb{Q}(\alpha)$, write $(3\alpha^2 + 2\alpha + 1)^{-1}$ in terms of the basis $1, \alpha, \alpha^2$.

5. Let A be an $n \times n$ matrix over the field F and F^n denote the vector space of column vectors of length n. Suppose A is idempotent, i.e., $A^2 = A$.
   (i) Prove that $F^n = U \oplus W$, where $A \cdot u = 0$, for all $u \in U$ and $A \cdot w = w$, for all $w \in W$.
   (ii) If $F = \mathbb{Z}_p$, how many idempotent 3×3 matrices are there?

6. Find the characteristic polynomial, the minimal polynomial, the rational canonical form and the Jordan canonical form for the matrix $A = \begin{pmatrix} 2 & 0 & 0 \\ 9 & 7 & 5 \\ -9 & -5 & -3 \end{pmatrix}$. Here we assume A has coefficients in a field of characteristic zero. How does your answer change if the entries of A belong to a field of positive characteristic?

ALGEBRA QUALIFYING EXAM: AUGUST 15, 2017

All answers must be fully justified to receive full credit. All problems are of equal value.

1. Let S_n denote the symmetric group on n letters.
   (i) For the k-cycle $\sigma = (a_1, \dots, a_k) \in S_n$ and $\tau \in S_n$, prove that $\tau\sigma\tau^{-1} = (\tau(a_1), \dots, \tau(a_k))$.
   (ii) Use part (i) to find the number of distinct conjugacy classes in S_5.

2. Let R be a unique factorization domain with quotient field K. Write $R[x]$ and $K[x]$ respectively, for the polynomial rings over R and K. Fix $0 \neq f(x) \in R[x]$. Let I denote the principal ideal $f(x)R[x]$ and J denote the ideal $f(x)K[x] \cap R[x]$. Prove that there exists $0 \neq a \in R$ such that $I = aJ$, where $aJ := \{aj \mid j \in J\}$.

3. Let $F \subseteq K$ be a simple algebraic extension of fields, i.e., $K = F(\alpha)$, for some $\alpha \in K$ algebraic over F, and write $\overline{F}$ to denote the algebraic closure of F. Let $E(K/F)$ denote the number of distinct field homomorphisms $\sigma : K \to \overline{F}$ fixing F.
   (i) Prove that $E(K/F) \leq [K : F]$, where $[K : F]$ denotes the degree of K over F.
   (ii) Give an example where $E(K/F) < [K : F]$.

4. Let $T : V \to W$ be a linear transformation of vector spaces over the field F, and assume $\dim(V) = \dim(W)$. Prove that there exist bases $\mathcal{B}$ and $\mathcal{B}'$ of V and W respectively, so that $[T]_{\mathcal{B}}^{\mathcal{B}'}$ is a diagonal matrix.

5. Let V be a finite dimensional inner product space over $\mathbb{R}$ and $v_1, \dots, v_n$ a basis for V. Let $c_1, \dots, c_n \in \mathbb{R}$. Prove that there exists $v \in V$ such that $\langle v, v_i \rangle = c_i$, for all $1 \leq i \leq n$.

6. Consider the matrix $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ -2 & 0 & -1 & -2 \\ 1 & -1 & 0 & 0 \end{bmatrix}$, with entries in $\mathbb{C}$. Find J, the Jordan canonical form for A and an invertible matrix P such that $J = P^{-1}AP$.

ALGEBRA QUALIFYING EXAM: JANUARY 11, 2016

All answers must be fully justified to receive full credit.

1. Recall that a group G is said to be *solvable*, if there exists a descending sequence of subgroups

$$G = G_0 \supset G_1 \supset \cdots \supset G_{n-1} \supset G_n = \{e\}$$

satisfying: (a) each G_{i+1} is a normal subgroup of G_i and (b) each factor group G_{i+1}/G_i is abelian. Prove that a finite group of order 105 is solvable.

2. Let F be a field and $f(x)$ an irreducible, separable polynomial over F. Write E for the splitting field of $f(x)$ over F, and suppose that α and $\alpha + 1$ are roots of $f(x)$.

(a) Prove that the characteristic of F is not zero.

(b) Prove that there exists a field L between F and E such that $[E : L]$ equals the characteristic of F.

3. Let K be a field and R a commutative integral domain containing K. Assume that R is a finite dimensional vector space over K. Prove that R is a field.

4. Set $A := \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ -3 & -2 & 0 & 2 \end{pmatrix}$ and regard A as a matrix with real coefficients. Find the rational canonical form B and the Jordan canonical form C of A. Give invertible matrices P and Q such that $P^{-1}AP = B$ and $Q^{-1}AQ = C$.

5. Let V be a finite dimensional vector space over the field F and $T : V \to V$ is a linear transformation. Suppose that $\chi_T(x)$, the characteristic polynomial of T, factors as $\chi_T(x) = p_1(x)^{f_1} \cdots p_r(x)^{f_r}$, where each $p_i(x)$ is irreducible over F. Prove that the minimal polynomial of T equals $p_1(x)^{e_1} \cdots p_r(x)^{e_r}$, where for each $1 \le i \le r$, e_i is the first exponent for which the kernel of $p_i(T)^{e_i}$ equals the kernel of $p_i(T)^{e_i+1}$.

6. Let V be a finite dimensional inner product space over $\mathbb{C}$ and $T : V \to V$ a linear transformation. Write T^* for the adjoint of T and let B denote an orthonormal basis for V.

(a) Prove that the matrix of T^* with respect B is the conjugate transpose of the matrix of T with respect to B.

(b) Let A denote the matrix of T with respect to B. What properties does A have if T is: (i) Hermetian or (ii) Unitary.

ALGEBRA QUALIFYING EXAM: AUGUST 16, 2016

All answers must be fully justified to receive full credit. All problems are of equal value.

1. Let R be a commutative ring with unity. Let $U(R)$ be the set of units in R.
   (a) a Show that $U(R)$ is an abelian group under multiplication.
   (b) Let $R = \mathbb{Z}[x]/(x^2)$. Show that $U(R) \cong \mathbb{Z} \times \mathbb{Z}_2$. Justify your answer carefully.

2. Recall that a splitting field for a polynomial $f(x)$ over a field F is a smallest (in terms of degree) extension K of F such that $f(x)$ factors completely in $K[x]$. Let $F = \mathbb{Z}_2$.
   (a) Prove that $K = F[t]/(t^3 + t + 1)$ is a field of order 8.
   (b) Prove that K is a splitting field for $f(x) = x^7 - 1$ over F.

3. Let G be a group and $Aut(G)$ be the group of automorphisms of G. Recall that for any $c \in G$, we have an automorphism of G given by $f_c(x) = c^{-1}xc$ (such an automorphism is called an inner automorphism of G). It is not hard to see that $Inn(G)$, the collection of inner automorphisms of G, is a subgroup of $Aut(G)$.
   (a) Show that $G/Z(G) \cong Inn(G)$. Here $Z(G)$ is the center of G.
   (b) Show that if $Aut(G)$ is cyclic, then G is abelian.
   (c) Find all prime numbers p such that there is a finite group G with $Aut(G) \cong \mathbb{Z}_p$. (Hint: show that if G is not $\mathbb{Z}_2$, then $Aut(G)$ must have an element of order 2).

4. Let A be an $n \times n$ matrix over the field F. Prove that the determinant of A equals $(-1)^n$ times the constant term of the characteristic polynomial of A.

5. Let V denote the complex vector space of complex polynomials having degree at most five, endowed with the usual inner product. In other words, for $f(x), g(x) \in V$, $\langle f(x), g(x) := \int_0^1 f(x)\overline{g(x)}\, dx$.
   (a) Find an orthonormal basis for the space W spanned by $x + x^2, x^2 + x^3, x^3 + x^4$.
   (b) Find (with proof) a self adjoint operator $T : W \to W$ that is not the identity map.

6. Let F be a field. For a vector space U over F, write U^* for the dual space of U. For vector spaces $V_1, \ldots, V_n$ over F prove that $(V_1 \oplus \cdots \oplus V_n)^*$ is isomorphic to $V_1^* \oplus \cdots \oplus V_n^*$.

ALGEBRA QUALIFYING EXAM: JANUARY 13, 2015

All answers must be fully justified to receive full credit.

1. Let G be a finite abelian group of odd order. Prove that for each $x \in G$, there exists a unique $y \in G$, such that $y^2 = x$.

2. Prove that a group of order $435 = 3 \cdot 5 \cdot 29$ must be abelian.

3. Let $k = \mathbb{Z}/2\mathbb{Z}$ and set $R := k[X, Y]$, the polynomial ring in two variables over k. Let $I \subseteq R$ be a proper ideal.
   (i) Explain why R/I is a vector space over k.
   (ii) Let q be a fixed power of two and suppose that $f_1, \dots, f_n$ is a set of generators for I. Set $I^{[q]}$ to be the ideal generated by $f_1^q, \dots, f_n^q$. Prove that $I^{[q]}$ is an ideal of R that does not depend upon the set of generators chosen.
   (iii) Compute the dimension of $R/I^{[q]}$ as a vectors space over k.

4. Find the minimal polynomial of $\sqrt{2} + \sqrt[3]{2}$ over $\mathbb{Q}$.

5. Let V be a vector space over the field F.
   (i) Define the dual space V^*.
   (ii) Prove that the canonical map $V \to V^{**}$ is an isomorphism when V is finite dimensional over F.
   (iii) Give an example to show that (ii) may fail if V is not finite dimensional.

6. Let A be an $n \times n$ matrix with entries in $\mathbb{R}$, the field of real numbers. Let $\phi_A : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation defined by $\phi_A(v) = A \cdot v$, for each column vector $v \in \mathbb{R}^n$. Set $W := \{v \in \mathbb{R}^n \mid \phi_A(v) = v\}$ and assume that $\dim \mathrm{Ker}(\phi_A) + \dim W = n$.
   (i) Give the minimal polynomial for A.
   (ii) Describe all possible Jordan canonical forms for A.
   (iii) Prove that if A is a symmetric matrix, then W is orthogonal to $\mathrm{Ker}(\phi_A)$. Assume that $\mathbb{R}^n$ is endowed with its standard inner product.

ALGEBRA QUALIFYING EXAM: AUGUST 18, 2015

All answers must be fully justified to receive full credit.

1. For a group G, we write $Z(G)$ to denote the *center* of G, i.e., the set of elements $x \in G$ such that $xg = gx$, for all $g \in G$. Note that the center of any group is a normal subgroup. Define subgroups $Z_i(G)$ inductively as follows: $Z_0(G) = \{e\}$. For $i \geq 0$, set $Z_{i+1}(G)$ to be the subgroup of G that is the pre-image of the center of the group $G/Z_i(G)$, i.e., $Z_{i+1}(G)/Z_i(G)$ is the center of $G/Z_i(G)$. A group G is said to be *nilpotent* if $Z_n(G) = G$, for some $n \geq 1$.
   (a) Show that each $Z_i(G)$ is a normal subgroup of G.
   (b) Suppose $|G| = p^r$, with p prime. Prove that G is a nilpotent group.
   (c) Give an example of a group G with a normal subgroup H such that H and G/H are nilpotent, but G is not nilpotent.

2. Let ϵ be a primitive sixth root of unity and δ be a primitive eighth root of unity. Prove that $\mathbb{Q}(\epsilon) \cap \mathbb{Q}(\delta) = \mathbb{Q}$.

3. Set $f(x) = x^3 + 2 \in \mathbb{Z}[x]$ and fix $\alpha \in \mathbb{C}$, a root of $f(x)$. Set $K := \mathbb{Q}(\alpha)$. Let R denote the subset of K consisting of elements of the form $a + b\alpha + c\alpha^2$, with $a, b, c \in \mathbb{Z}$.
   (a) Show that R is a subring of K.
   (b) Show that for every non-zero ideal $I \subseteq \mathbb{Z}$, $IR \cap \mathbb{Z} = I$. Here, IR denotes the ideal of R generated by I.
   (c) Show that for every non-zero ideal $J \subseteq R$, $J \cap \mathbb{Z} \neq 0$.
   (d) Show that every non-zero prime ideal in R is a maximal ideal.

4. Let V denote the space of $n \times n$ matrices over a field F. Fix $A \in V$ and define $T_A : V \to V$ by $T_A(B) = AB$, for all $B \in V$. Show that the minimal polynomial for T_A equals the minimal polynomial for A.

5. Let A and B be $n \times n$ commuting matrices over $\mathbb{C}$.
   (a) Prove that A and B have a common eigenvector.
   (b) Assume that B has n distinct eigenvalues. Prove that A is diagonalizable.

6. Let A be the $n \times n$ matrix over $\mathbb{C}$ all of whose entries equal 1.
   (a) Find the minimal polynomial of A.
   (b) Find the Jordan canonical form of A.
   (c) Let B denote the $n \times n$ matrix over $\mathbb{C}$ whose diagonal entries are 0 and all other entries are 1. Find the Jordan canonical form of B.

ALGEBRA QUALIFYING EXAM : JANUARY 13, 2014

Show all work to receive full credit. When in doubt, it is better to show more work than less.

1. Let G be a finite group. Recall that for $x \in G$, the *centralizer of* x, denoted $C_G(x)$ is the set of elements in G commutig with x.
   (i) Prove that for any $x \in G$, $C_G(x)$ is a subgroup of G whose index in G equals the number of elements of G conjugate to x. (5 points)
   (ii) Assume that the order of G is odd. Prove that for any non-identity element $x \in G$, x and x^{-1} are not conjugate in G. (10 points).

2. Let G be a group, not necessarily finite, and $H \subseteq G$ a subgroup with $[G : H] = n$. Here we are writing $[G : H]$ to denote the index of H in G. Prove that there exists a normal subgroup K of G with $K \subseteq H$ and $[G : K] \leq n!$. (15 points)

3. Let $p > 0$ be a prime number and X, Y independent indeterminates over the field $\mathbb{Z}_p$. Let F and K denote the rational function fields $\mathbb{Z}_p(X^p, Y^p)$ and $\mathbb{Z}_p(X, Y)$, respectively.
   (i) Calculate $[K : F]$, the degree of K over F. (10 points)
   (ii) Prove that K is not a simple extension of F, i.e., there does not exist $\alpha \in K$ such that $K = F(\alpha)$. (10 points)

4. For this problem you may assume the fact that any proper ideal in a commutative ring is contained in a maximal ideal.
   (i) Let R be a commutative ring and $M_1, M_2 \subseteq R$ distinct maximal ideals. Show that for any $n \geq 1$, $R = M_1^n + M_2^n$. Recall, that for any ideal $I \subseteq R$, I^n is the ideal of R consisting of all finite sums of n-fold products of elements of I. (10 points)
   (ii) Now let $\mathbb{R}[X, Y]$ denote the set of real polynomials in two variables. Fix $n \geq 1$ and two distinct points $P_1 := (\alpha_1, \beta_1)$ and $P_2 := (\alpha_2, \beta_2)$ in $\mathbb{R}^2$. Let $f(X, Y) \in \mathbb{R}[X, Y]$. Prove that there exist $g(X, Y), h(X, Y)$ such that :
   (a) $f(X, Y) = g(X, Y) + h(X, Y)$
   (b) $g(X, Y)$ and all of its partial derivatives of order less than n vanish at P_1
   (c) $h(X, Y)$ and all of its partial derivatives of order less than n vanish at P_2.
   (10 points)

5. Let V and W be vector spaces over the field F of dimensions n and m respectively. Let $\mathcal{B}_V$ and $\mathcal{B}'_V$ be bases of V and $\mathcal{B}_W$ and $\mathcal{B}'_W$ be bases for W. Let $T : V \to W$ be a linear transformation. Let A denote the matrix of T with respect to $\mathcal{B}_V$ and $\mathcal{B}_W$ and A' denote the matrix of T with respect to $\mathcal{B}'_V$ and $\mathcal{B}'_W$. First state, then prove, the relationship between A and A' in terms of invertible matrices. Be very specific in your statement. Then restate (but not re-prove) this relationship in the special case that $W = V$. (15 points)

6. Let V be a finite dimensional vector space over $\mathbb{R}$ and $T : V \to V$ a linear operator on V. Assume that the dimension of V is at least four. Give a proof of, or provide a counter-example (with justification) to, the following statement. *There exists a basis $\mathcal{B}$ of V such that the matrix of T with respect to $\mathcal{B}$ is block diagonal, with each block either a 1×1 or a 2×2 matrix.* (15 points)

ALGEBRA QUALIFYING EXAM: AUGUST 19, 2014

Show all work to receive full credit. When in doubt, it is better to show more work than less.

1. Let $W_1 \subseteq V_1$ and $W_2 \subseteq V_2$ be vector spaces over the field F. Set $\mathcal{U} := \{T \in \mathcal{L}(V_1, V_2) \mid T(W_1) \subseteq W_2\}$, where $\mathcal{L}(V_1, V_2)$ denotes the space of linear transformations from V_1 to V_2. Note that $\mathcal{U}$ is a subspace of $\mathcal{L}(V_1, V_2)$.

   (i) Show that there exists a surjective linear transformation $\phi : \mathcal{U} \to \mathcal{L}(V_1/W_1, V_2/W_2)$. (8 points)

   (ii) Identitfy (with proof) the kernel of ϕ. (4 points)

   (iii) Assume that V_1 and V_2 are finite dimensional over F. Find a formula for the dimension of $\mathcal{U}$. (4 points)

2. Let V denote the vector space of 2×2 matrices over the field of complex numbers and set $A := \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$. Let $T_A : V \to V$ be the linear transformation given by $T_A(B) = A \cdot B$, for all $B \in V$. Find the Jordan canonical form for T_A and find a basis $\mathcal{B}$ for V such that the matrix of T_A with respect to $\mathcal{B}$ is in Jordan canonical form. (18 points)

3. Let V be a finite dimensional vector space over the field F and $T : V \to V$ a linear transformation.

   (i) For $v \in V$, let $\mu_{T,v}(X)$ denote the monic polynomial of least degree such that $\mu_{T,v}(T)(v) = 0$. Prove that v is a cyclic vector for V with respect to T if and only if the degree of $\mu_{T,v}(X)$ equals the dimension of V. (5 points)

   (ii) Suppose $V = W_1 \oplus W_2$, for T-invariant subspaces $W_1, W_2 \subseteq V$. Write $\mu_1(X)$ for the minimal polynomial of $T|_{W_1}$ and $\mu_2(X)$ for the minimal polynomial of $T|_{W_2}$ and suppose that $\mu_1(X)$ and $\mu_2(X)$ are relatively prime. For $w_i \in W_i$, prove that $v := w_1 + w_2$ is a cyclic vector for V with respect to T if and only if w_i is a cyclic vector for W_i with respect to $T|_{W_i}$, for $i = 1, 2$. (8 points)

   (iii) Give a specific example where the conclusion of (i) fails in case $\mu_1(X)$ and $\mu_2(X)$ are not relatively prime. (5 points)

4. Let $p > 5$ be a prime that is not congruent to 1 modulo 5. Prove that any group of order $15p$ contains a subgroup of order $5p$. State carefully any theorem you use to prove this result. (16 points)

5. Let R be a principal ideal domain. For $f, g \in R$, show that $f^{1000}g^{1014}$ belongs to the ideal generated by f^{2014} and g^{2014}. (16 points)

6. Show that $X^5 - 2$ is irreducible over the field $\mathbb{Z}_{31}$. (16 points)

ALGEBRA QUALIFYING EXAM : JANUARY 14, 2013

Show all work to receive full credit. When in doubt, it is better to show more work than less.

1. Exhibit (with proof) an element of largest order in S_{12}, the symmetric group on 12 letters.

2. Let R be a commutative ring. An element $a \in R$ is said to be *nilpotent* if $a^n = 0$, for some n.
   (i) Prove that the set of nilpotent elements form an ideal. This ideal is called the *nilradical* of R.
   (ii) Give a detailed description (with proof) of the nilradical of $\mathbb{Z}_9[X]$, the polynomial ring with coefficients in $\mathbb{Z}_9$.

3. Let $F \subseteq K$ be fields and $\alpha \in K \backslash F$. Recall that the subfield of K generated over F by α is the field $F(\alpha)$ obtained by taking the intersection of all subfields of K containing F and α.
   (i) Suppose that α is *not* algebraic over F. Prove that every element of $F(\alpha)$ is of the form $g(\alpha) \cdot h(\alpha)^{-1}$, for some $g(X), h(X) \in F[X]$.
   (ii) Let $F \subseteq \mathbb{R}$ be a subfield of the real numbers that is maximal with respect to the property of not containing π. (Such a field exists by Zorn's Lemma.) Prove that π is algebraic over F.

4. Let V be a vector space of dimension n over the field F and $T : V \to V$ be a linear transformation. Write $\mathcal{L}(V)$ for the vector space of linear transformations of V and $\mathcal{C}_T$ for the set of $S \in \mathcal{L}(V)$ such that $TS = ST$.
   (i) Prove that $\mathcal{C}_T$ is a subspace of $\mathcal{L}(V)$.
   (ii) Assume that T has a cyclic vector, i.e., there exists $v \in V$ such that v, T(v)$, \dots, T^{n-1}(v)$ form a basis for V. Prove that if $S \in \mathcal{C}_T$, then $S = p(T)$, for some polynomial $p(X) \in F[X]$.
   (iii) For T as in (ii), what is the dimension of $\mathcal{C}_T$?
   (iv) Call a transformation in $\mathcal{L}(V)$ *nice* if it satisfies the conclusion of (ii). Classify all nice linear transformations in the case $n = 3$ and $F = \mathbb{C}$.

Remark : In problem 4, if you prefer, you may work with matrices.

5. Let T be a linear transformation on $\mathbb{R}^n$. Use the fact that any polynomial with real coefficients factors into quadratic and linear polynomials to prove that T has a non-zero invariant subspace of dimension at most 2.

6. For each 4×4 matrix A over $\mathbb{R}$ with minimal polynomial $f(x) = (X^2 + 1)^2$:
   (i) Find the rational canonical H form of A.
   (ii) Now regard A as a matrix over $\mathbb{C}$. Find the rational canonical form B.
   (iii) Are H and B similar over $\mathbb{C}$? Justify your answer.
   (iv) Repeat (i)-(ii) for 5×5 matrices.

ALGEBRA QUALIFYING EXAM : AUGUST 20, 2013

Show all work to receive full credit. When in doubt, it is better to show more work than less.

1. Let G be a group with subgroups H and K. Set $HK := \{h \cdot k \mid h \in H \text{ and } k \in K\}$.
   (a) Prove that HK is a subgroup if and only if $HK = KH$. Conclude that if either H or K is a normal subgroup of G, then HK is automatically a subgroup of G. (8 Points)
   (b) For a finite subset C of G, let $|C|$ denote the number of elements in C. Prove that if H and K are finite, then
$$|HK| = \frac{|H| \cdot |K|}{|H \cap K|} \text{ (8 points).}$$

2. Let R be an integral domain and $\{P_n\}_{n=1}^{\infty}$ an infinite collection of prime ideals.
   (a) Prove that if $P_1 \supseteq P_2 \supseteq P_3 \supseteq \cdots$, then $\bigcap_{n=1}^{\infty} P_n$ is a prime ideal. (8 points)
   (b) Give an example to show that the conclusion of part (a) fails if the given collection of primes does not form a descending chain. (8 Points)

3. Let $F \subseteq K$ be fields and $f(X)$ be an irreducible polynomial with coefficients in F. Let $\alpha \in K$ be a root of $f(X)$.
   (a) Suppose that $\sigma : K \to K$ is an automorphism of K fixing F. Prove that $\beta := \sigma(\alpha)$ is a root of $f(X)$. (6 points)
   (b) Prove that if $\beta \in K$ is also a root of $f(X)$, then there exists an isomorphism of fields $\tau : F(\alpha) \to F(\beta)$ that fixes F and takes α to β. (10 points)

4. Let $f(X) := X^3 - 2 \in \mathbb{Q}[X]$.
   (a) Prove that $f(X)$ is irreducible over $\mathbb{Q}$. (3 points)
   (b) Describe the splitting field K of $f(X)$ over $\mathbb{Q}$. (3 points)
   (c) Use problem 3 to prove that there exists an automorphism $\sigma : K \to K$ that cyclically permutes the roots of $f(X)$. (6 points)
   (d) For σ in (c), describe the set $\{\gamma \in K \mid \sigma(\gamma) = \gamma\}$. (Hint: the set of elements in question is a subfield of K.) (6 points)

5. Let V be a finite dimensional vector space over a field and $T : V \to V$ a linear transformation. Show that there exists $n \geq 1$ such that $V = kernel(T^n) \bigoplus image(T^n)$. (16 points)

6. Recall that a square matrix A over a field F is said to be *skew symmetric* if $A^t = -A$. Let $\mathcal{V}$ denote the space of 3×3 skew-symmetric matrices over $\mathbb{C}$ and $M_3(\mathbb{C})$ the space of all 3×3 matrices over $\mathbb{C}$. For a fixed matrix $A \in M_3(\mathbb{C})$, let $\phi_A : \mathcal{V} \to \mathcal{V}$ denote the linear transformation $\phi_A(U) = AUA^t$, for all $U \in \mathcal{V}$.
   (a) Show that a matrix in $\mathcal{V}$ is either nilpotent or has three distinct eigenvalues. (4 points)
   (b) Fix a basis $\mathcal{B}$ for $\mathcal{V}$. For $A \in M_3(\mathbb{C})$, let $\widehat{A}$ denote the matrix of ϕ_A with respect to $\mathcal{B}$. Prove that $\widehat{AB} = \widehat{A}\widehat{B}$, for all $A, B \in M_3(\mathbb{C})$. (6 points)
   (c) Suppose $A \in M_3(\mathbb{C})$ has three distinct eigenvalues. Write the eigenvalues of $\widehat{A}$ in terms of the eigenvalues of A. (Hint : Part (c) should be helpful.) (8 points)

PRELIMINARY VERSION OF ALGEBRA QUALIFYING EXAM : AUGUST 14, 2012

You must show all work on each problem to receive full credit. All problems are of equal value.

1. Consider the abelian group $G := \mathbb{Z} \times \mathbb{Z}_{30}$ and H the subgroup generated by the element (5,3). Describe the group G/H. You must justify your answer rigorously.

2. Prove that $F := \mathbb{Z}_2[X]/(X^3 - X + 1)$ is a field. How many elements does F have ? For $f(X) \in \mathbb{Z}_2[X]$, let $[f(X)]$ denote the class of $f(X)$ in F. Find the multiplicative inverse of $[X^2 + 1]$ in F.

3. Let $R \subseteq T$ be integral domains and suppose that $a \in T$ satisfies a monic polynomial of degree d with coefficients in R. Let S denote the intersection of all subrings of T containing R and a.

(a) Show that $S = \{r_0 + r_1 a + \cdots + r_{d-1}a^{d-1} \mid r_0, \ldots, r_{d-1} \in R\}$.

(b) Prove that if Q is a maximal ideal of R, then there are at most d maximal ideals $P \subseteq S$ with $P \cap R = Q$.

4. Let A be an $n \times n$ matrix over $\mathbb{R}$ such that A^2 is a non-zero scalar matrix. Prove that A is a symmetric matrix if and only if A is a scalar multiple of an orthogonal matrix.

5. For $A := \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$, find a 3×3 orthogonal matrix P such that PAP^{-1} is a diagonal matrix.

6. Let $\mathbb{F}_2 = \{0, 1\}$ denote the field with two elements. Take $a_1, \ldots, a_n \in \mathbb{F}_2$ and consider the following $n \times n$ matrix

$$\sigma = \begin{bmatrix} 0 & 0 & \cdots & 0 & a_1 \\ 1 & 0 & \cdots & 0 & a_2 \\ 0 & 1 & \cdots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_n \end{bmatrix}.$$

(A) Prove that σ has no eigenvalues if and only if $a_1 = 1$ and $a_2 + \cdots + a_n = 1$.

(B) Assume that σ is invertible and let $v \in \mathbb{F}_2^n$ be a non-zero vector. Set $S_v := \{\sigma^j(v) \mid j \geq 0\}$. Show that that there exists $r \geq 1$ such that $S_v = \{v, \sigma(v), \ldots, \sigma^{r-1}(v)\}$ and $\sigma^r(w) = w$, for all $w \in S_v$.

(C) Assume that $p := 2^n - 1$ is a prime number. Show that if the cyclic group $\langle\sigma\rangle$ has order p, then $a_1 = 1$ and $a_2 + \cdots + a_n = 1$. (Hint : Consider the sets S_v for $v \in \mathbb{F}_2^n$.)

Algebra Qualifying Exam : January 2011

You must show all work to receive full credit. When in doubt, it is better to show more work than less work. All problems are of equal value.

1. Let G be a finite group of order three or more. Prove that G has at least two distinct automorphisms.

2. Let R be a commutative ring and $I, J \subseteq R$ ideals. Set

$$(I : J) := \{r \in R \mid r \cdot j \in I \text{ for all } j \in J\}.$$

Prove that $(I : J)$ is an ideal of R. Now suppose that R is a unique factorization domain. Prove that for all principal ideals I and J, $(I : J)$ is a principal ideal.

3. Let F be the smallest subfield of the complex numbers containing all fourth roots of 5. Give a concrete description of F. Give a detailed proof that for any positive prime number p, F does not contain any cube root of p.

4. Let $A := \begin{pmatrix} 3 & -4 & -4 \\ -1 & 3 & 2 \\ 2 & -4 & -3 \end{pmatrix}$. Find an invertible 3×3 matrix P such that PAP^{-1} is the Jordan canonical form of A.

5. Let T be a linear operator on $\mathbb{R}^n$ with adjoint T^*. Prove that $Ker(T) = (Im(T^*))^{\perp}$.

6. For $n \geq 0$, let V be the set of infinite sequences of real numbers $\{a_n\}$ such that for all $n \geq 1$, $a_{n+1} = a_n + a_{n-1}$. Prove V is a vector space. Find the dimension of V and a basis of V.

Algebra Qualifying Exam : August 2011

Please work the following problems. You must show all work to receive full credit. All problems are of equal value. $\mathbb{Z}$ denotes the ring of integers and $\mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ denote the rational, real, and complex numbers respectively.

1. Prove that the center of a non-abelian group of order 21 must be just the identity.

2. How many distinct ideals does the ring $R = \mathbb{Z}[i]/(6)$ have? (Here, $i^2 = -1$).

3. Let $F := \mathbb{Z}_2$ denote the field with two elements and $f(x)$ the polynomial $x^{12} + x^4 + 1$ with coefficients in F. Let α be a root of $f(x)$.

(a) Find with proof the number of elements in the field $F(\alpha)$.

(b) Does $F(\alpha)$ contain all of the roots to $f(x)$? Justify your answer.

4. Let V be a finite dimensional over $\mathbb{R}$, and let T be a linear operator on V without an eigenvalue. Prove that every subspace of V which is invariant under T has even dimension.

5. Let V be a finite dimensional vector space over $\mathbb{C}$, and let $v_1, .., v_m$ be nonzero vectors of V. Prove there is a linear functional $f : V \to \mathbb{C}$ such that $f(v_i) \neq 0$ for all $i = 1, ..., m$.

6. Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. Discuss the difference - if any - between the rational canonical form of A over $\mathbb{Q}$ and the rational canonical form of A over $\mathbb{Z}_3$.

Qualifying Exam **Algebra** January, 2010

Each question is given equal weight in the grading. $\mathbb{Q}$ denotes the rational numbers, $\mathbb{R}$ the reals, and $\mathbb{C}$ the complex numbers. Fields should not be assumed to be the reals or complex numbers unless specifically stated. You should justify all assertions to receive full credit. Demonstrating an in-depth understanding of the material is the most important factor.

1. Exhibit an explicit isomorphism between S_3, the symmetric group on 3 elements, and the group of all invertible 2 by 2 matrices over the field F_2 with 2 elements.

2. Let $\phi : \mathbb{C}[x,y] \to \mathbb{C}[t,t^{-1}]$ be the $\mathbb{C}$-algebra homomorphism defined by $\phi(f(x,y)) = f(t,t^{-1})$. Find (with proof) a polynomial $g(x,y)$ such that the kernel of ϕ is the principal ideal in $\mathbb{C}[x,y]$ generated by g. Prove that there is an isomorphism of $\mathbb{C}$-algebras from $\mathbb{C}[x,y]/(x^2+y^2-1)$ to $\mathbb{C}[t,t^{-1}]$.

3. Find (with proof) the minimal polynomial of $\cos(\frac{2\pi}{5})$ over $\mathbb{Q}$.

4. Let F_5 be the field having 5 elements, and let V be a vector space of dimension n over F_5. Find the cardinality of V. Let Λ denote the set of all ordered n-tuples, $(v_1,...,v_n)$, where $v_1,...,v_n$ form a basis of V. What is the cardinality of Λ? Use this to find the number of 3 by 3 invertible matrices over F_5.

5 Consider the matrix

$$A = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & -1 \\ 1 & 0 & 2 & 0 \\ -4 & 1 & -2 & 2 \end{pmatrix}$$

over the real numbers. Find (with justification) the rational canonical form of A.

6. Let A be an n by n matrix over the real numbers. Prove there exists an n by n real matrix B such that $A = B^T B$ if and only if A is symmetric and all the eigenvalues of A are non-negative. Here B^T is the transpose of B. (Hint: first recall what you know concerning diagonalizing A.)

Algebra Qualifying Examination : August 2010

You must show all work to receive full credit. When in doubt, show more work, not less. Each problem is given an equal weight in the grading. Demonstrating an in-depth understanding of the material is the most important factor.

1. Prove that any group of order 30 has a cyclic subgroup of order 15.

2. Recall that if J is an ideal in the commutative ring R, we write $J = (a_1, \ldots, a_n)$ if J can be generated by the elements $a_1, \ldots, a_n \in R$. Now let $R := \mathbb{Z}[X]$, $M_1 := (3, X^2 + X + 2)$ and $M_2 := (2, X^2 + X + 1)$.

   (a) Show that M_1 and M_2 are maximal ideals of R.

   (b) Find a set of generators for the ideal $M_1 \cap M_2$.

3. For each of the following polynomials, find the degree of its splitting field over $\mathbb{Q}$: (a) $X^4 + X^2 + 1$ and (b) $X^3 + 2X + 3$.

4. For the complex matrix $A := \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & i & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & i \end{pmatrix}$, find an invertible complex matrix P such that PAP^{-1} is in Jordan Canonical Form.

5. Let V be a three-dimensional vector space over the rational numbers and T a linear transformation on V. Suppose that there are three vectors $u, v, w \in V$ with u nonzero such that $T(u) = v$, $T(v) = w$, and $T(w) = u + v$. Prove u, v, w form a basis of V.

6. Let V be a finite dimension vector space and T, S linear transformations on V. Prove or give a counterexample to the following two statements:

   (a) Every eigenvector of TS is also an eigenvector of ST.

   (b) Every eigenvalue of TS is also an eigenvalue of ST.

ALGEBRA QUALIFYING EXAMINATION : AUGUST 2009

You must show all work to receive full credit. When in doubt, it is better to show more work than less work. All problems are of equal value.

1. Prove that, up to isomorphism, there is only one finite group of order 77.

2. Let R be a commutative ring. Recall that the *Jacobson radical* of R is the intersection of all of the maximal ideals of R. (a) Prove that the Jacobson radical of R equals the set $\{a \in R \mid 1 + ax \text{ is a unit for all } x \text{ in } R\}$. You may use (without proof) the fact that any proper ideal of R is contained in a maximal ideal. (b) Find, with proof, the Jacobson radical of the polynomial ring $\mathbb{Z}[X]$ in one variable over $\mathbb{Z}$.

3. Let F be the smallest subfield of the complex numbers containing all of the fifth roots of 2. Find, with proof, a basis for F over $\mathbb{Q}$.

4. Let A be an $n \times n$ matrix over the field F such that the rank of A is $r > 0$. Prove that there exist invertible matrices P, Q with entries in F such that PAQ has r 1's down the diagonal and zeroes elsewhere.

5. Let $f(x) = (x^2 + 1) \cdot (x^2 + x + 1) \in \mathbb{R}[x]$.

(i) Prove that there is no 5×5 matrix with entries in $\mathbb{R}$ that has $f(x)$ as its minimal polynomial.

(ii) Find two non-similar 6×6 matrices over $\mathbb{R}$ that have $f(x)$ as their minimal polynomial. (Recall that matrices A and B are *similar* if there exists an invertible matrix U such that $B = UAU^{-1}$.)

(iii) Find the Jordan canonical forms for your matrices in (ii).

6. Let V be the set of complex numbers regarded as a vector space over $\mathbb{R}$. The identity $< z_1, z_2 > := Re(z_1 \cdot \overline{z_2})$ defines an inner product on V. For each $w \in V$, $T_w(z) := w \cdot z$, for all $z \in V$, defines a linear operator on V.

(i) What is the determinant of T_w ?

(ii) For which complex numbers w is T_w self-adjoint ?

(iii) For which complex numbers w is T_w a unitary operator ?

(iv) Find a unitary operator on V that is not T_w for any $w \in V$.

Qualifying Exam **Algebra** January, 2009

Each question is given equal weight in the grading. $\mathbb{Q}$ denotes the rational numbers, $\mathbb{R}$ the reals, and $\mathbb{C}$ the complex numbers. Fields should not be assumed to be the reals or complex numbers unless specifically stated. You should justify all assertions to receive full credit. Demonstrating an in-depth understanding of the material is the most important factor.

1. Prove (without using the structure theorem for finite abelian groups) that the direct product of a cyclic group of order r with a cyclic group of order s is isomorphic to the direct product of a cyclic group of order $m = \operatorname{lcm}(r, s)$ with a cyclic group of order $n = \gcd(r, s)$.

2. Let $R = \mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ be the Gaussian integers ($i^2 = -1$). Let I be the principal ideal in R generated by $i - 2$. Prove that $R/I \cong \mathbb{Z}/5\mathbb{Z}$.

3. Let V be a finite dimensional inner product space, and let $f \in V^*$ be a linear functional. Prove that there is an element $w \in V$ such that for all $v \in V$, $f(v) = (v|w)$.

4. Let $V = \mathbb{C}^3$, and let $T : V \to V$ be the linear operator with matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

with the standard basis of V. Prove that T has no cyclic vector. Give the Jordan and rational normal forms for T.

5. Let A be a 6 by 6 matrix over the complex numbers. Suppose that the minimal polynomial $p(x)$ of A generates the ideal in $\mathbb{C}[x]$ generated by the polynomials $f = (x^2 + 1)(x - 1)^3 x^2$ and $g = (x - 1)^2 x^3 (x + 1)^2$. What are the possible Jordan normal forms for A?

6. Let F be a field, and let E be a field containing F. Suppose that $F \subset L_1 \subset E$ and $F \subset L_2 \subset E$ are subfields of E containing F. Let L be the smallest subfield of E containing both L_1 and L_2. Prove that

$$[L : F] \leq [L_1 : F][L_2 : F].$$

When does equality hold?

Qualifying Exam: August 2008

I. What we are looking for is whether you have understood some important concepts and whether you are able to apply the general theory in particular situations. We also want to test your computational skills.

II. Give as many details as possible. If you are invoking a theorem to prove something you must state that theorem precisely.

III. **Q**, **R** and **C** stand for the field of rational numbers, real numbers and complex numbers respectively.

(1) Show that a finite cyclic group of order n has exactly one subgroup of each order d dividing n, and that these are all the subgroups it has. (15 points)

(2)

(a) Find the number of irreducible quadratic polynomials in $\mathbf{Z}_p[x]$, where p is a prime. (10 points)

(b) Show that for p a prime, the polynomial $x^p + a$ in $\mathbf{Z}_p[x]$ is not irreducible for any $a \in \mathbf{Z}_p$. (10 points)

(3)

(a) Find all fields that are finite extensions of $\mathbf{R}$. (10 points)

(You may assume that $\mathbf{C}$ is algebraically closed.)

(b) Prove that the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$ is not a finite extension of $\mathbf{Q}$. (10 points)

(4) Let $A = \begin{bmatrix} 3 & 2 & -3 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ be a 3×3 matirx.

(a) Find the characteristic polynomial and eigenvalues of A. (5 points)

(b) Find the Jordan canonical form J. (5 points)

(c) Find a matrix P so that $J = P^{-1}AP$. (10 points)

(5) Let $\boldsymbol{T} : \boldsymbol{C}^n \to \boldsymbol{C}^n$ be a linear transformation. We say $\mathbf{v} \in \boldsymbol{C}^n$ is a generalized eigenvector of T with corresponding eigenvalue λ if $\mathbf{v} \neq \mathbf{0}$ and $(T - \lambda I)^k(\mathbf{v}) = \mathbf{0}$ for some positive integer k. Define the generalized $\lambda-$eigenspace

$$E(\lambda) = \{\mathbf{v} \in \boldsymbol{C}^n | (T - \lambda I)^k(\mathbf{v}) = \mathbf{0} \text{ for some positive integer } k\}$$

(a) Prove that $E(\lambda)$ is a subspace of $\boldsymbol{C}^n$. (5 points)

(b) Suppose $\lambda_1, ..., \lambda_k$ are distinct scalars and $\mathbf{v}_1, ..., \mathbf{v}_k$ are generalized eigenvectors of T with corresponding eigenvalues $\lambda_1, ..., \lambda_k$ respectively Prove that $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$is a linearly independent set. (hint: first show this: Suppose $T(\mathbf{w}) = \lambda\mathbf{w}$. Prove that $(T - \mu I)^k(\mathbf{w}) = (\lambda - \mu)^k\mathbf{w})$ (10 points)

(6) Suppose that $\mathbf{v}_1, ..., \mathbf{v}_k \in \mathbf{R}^n$ are orthonormal and that for every $\mathbf{x} \in \mathbf{R}^n$ we have

$$\| \mathbf{x} \|^2 = (\mathbf{x} \cdot \mathbf{v}_1)^2 + ... + (\mathbf{x} \cdot \mathbf{v}_k)^2.$$

Prove that $k = n$ and deduce that $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ is an orthonormal basis for $\mathbf{R}^n$. (10 points)

Qualifying Exam (May 2008)

Answers, even if correct without relevant details will fetch very few points.

In the questions below: R denotes real numbers and Q denotes rational numbers.

(1) (a) Show that if G is nonabelian, then the factor group $\frac{G}{Z(G)}$ is not cyclic. ($Z(G)$ is the center of G)

(b) Show that a nonabelian group G of order pq where p and q are primes has a trivial center.

(2) Show that a nonconstant polynomial in $\mathbf{C}[x]$ has a zero in $\mathbf{C}$ if and only if the following is true: Let $f_1(x), ..., f_r(x) \in \mathbf{C}[x]$ and suppose that for every $\alpha \in \mathbf{C}$ that is a zero of all r of these polynomials is also a zero of a polynomial $g(x) \in \mathbf{C}[x]$. Then some power of $g(x)$ is in the smallest ideal of $\mathbf{C}[x]$ that contains the r polynomials $f_1(x), ..., f_r(x)$.

(3) What degree field extensions can we obtain by successively adjoining to a field $\mathbf{F}$ a square root of an element of $\mathbf{F}$ not a square in $\mathbf{F}$, then square root of some nonsquare in this new field, and so on? Argue from this that a zero $x^{14} - 3x^2 + 12$ over $\mathbf{Q}$ can never expressed as a rational function of square roots of rational functions of square roots, and so on, of elements of $\mathbf{Q}$.

(4) Let A and B be 2×2 matrices with integer entries such that A, $A + B$, $A + 2B$, $A + 3B$ and $A + 4B$ are all invertible matrices whose inverses have integer entries. Prove that $A + 5B$ is invertible and that its inverse has integer entries.

(**Hint:** Look at the function $f(t) = \det(A + tB)$ for different values of t)

(5) Let A be an $n \times n$ matrix all of whose eigenvalues are real numbers. Prove that there is a basis for $\mathbf{R}^n$ with respect to which the matrix for A becomes upper triangular.

(6) We say an $n \times n$ matrix N with real entries is nilpotent if $N^r = 0$ for some positive integer r.

(a) Show that 0 is the only eigenvalue of N.

Prove directly, without using the Jordan form, the following:

(b) Suppose $N^n = 0$ and $N^{n-1} \neq 0$. Prove that there is a basis $\{v_1, ... v_n\}$ for $\mathbf{R}^n$ with respect to which the matrix for N becomes

$$\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix}$$

(**Hint**: Might be helpful to show that the nullspace $\mathbf{N}(N)$ is one dimensional)

Qualifying Exam **Algebra** January, 2008

Each question is given equal weight in the grading. $\mathbb{Q}$ denotes the rational numbers, $\mathbb{R}$ the reals, and $\mathbb{C}$ the complex numbers. Fields should not be assumed to be the reals or complex numbers unless specifically stated. You should justify all assertions to receive full credit. Demonstrating an in-depth understanding of the material is the most important factor.

1.

(a) Let G be an abelian group. Let $a, b \in G$. Prove that the order of ab is the least common multiple of the orders of a and b if the intersection of the subgroup $A = < a >$ and the subgroup $B = < b >$ is the identity.

(b) Prove that the alternating group A_5 has exactly five subgroups of order 4.

2. Let k be a field, and let $f(x)$ and $g(x)$ be irreducible polynomials in the polynomial ring $k[x]$. Assume that the degrees of $f(x)$ and $g(x)$ are relatively prime. Let E be an extension field of k, and assume that there is a root α of $g(x)$ with $\alpha \in E$. Prove that $f(x)$ is still irreducible, considered in the polynomial ring $k(\alpha)[x]$.

3. Let R be a commutative integral domain containing a field k. Assume that R is finite dimensional as a vector space over k. Prove that R is a field.

4. Let
$$A = \begin{pmatrix} 0 & 3 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 10 & 0 & -3 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$
State the Jordan canonical form theorem, and find the Jordan canonical form for the matrix A, considered as a matrix with complex entries.

5. Let A be an n by n matrix over $\mathbb{C}$. Assume that A has distinct eigenvalues. Let W be the subspace of complex n by n matrices which commute with A. Prove that W has dimension at least n.

6. Let A be an n by n real symmetric matrix such that $T^k = I$ for some k, where I is the identity matrix. Prove that $T^2 = I$.

Qualifying Exam **Algebra** August, 2007

Each question is given equal weight in the grading. $\mathbb{Z}$ denotes the integers, $\mathbb{Q}$ denotes the rational numbers, $\mathbb{R}$ the reals, and $\mathbb{C}$ the complex numbers. Fields should not be assumed to be the reals or complex numbers unless specifically stated. You should justify all assertions to receive full credit. Although partial credit may be given, demonstrating an in-depth understanding of the material is the most important factor.

1. Let $f(x) \in \mathbb{Z}[x]$ be a monic polynomial. Let $\alpha \in \mathbb{C}$. Assume that both α and $3 \cdot \alpha$ are roots of $f(x)$. Prove that $f(0)$ is divisible by 3. (Hint: compare the minimal polynomials of α and $3 \cdot \alpha$ over $\mathbb{Q}$.)

2. Show that the degree of $\mathbb{Q}(\sqrt{5}, i)$ over $\mathbb{Q}$ is 4, where $i^2 = -1$. Find (with proof) an element $\alpha \in \mathbb{Q}(\sqrt{5}, i)$ such that $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{5}, i)$.

3. A group G is said to have *exponent* k if $g^k = e$ for all $g \in G$. Let n be a positive integer and let $\mathbb{Z}_n^*$ be the multiplicative group of units in the ring $\mathbb{Z}_n$. Find all values of n such that $\mathbb{Z}_n^*$ has exponent 2.

4. Let I be the ideal in $\mathbb{Z}[x]$ generated by the two polynomials $f = x^2 + 3x + 12$ and $g = x^2 + 3x + 82$. Find five maximal ideals containing I. Be sure to prove the ideals are maximal.

5. Let $T : V \to V$ be a linear transformation of an n-dimensional vector space over the complex numbers. Assume that the minimal polynomial of T is $(x-1)^n$. Find the Jordan canonical form of the operator $S = T^2$.

6. Let A be an n by n matrix over the real numbers. Prove that A is invertible if and only if the identity matrix I is in the $\mathbb{R}$-span of the matrices $A, A^2, A^3, \ldots, A^n$.

Qualifying Exam **Algebra** May, 2007

Each question is given equal weight in the grading. $\mathbb{Q}$ denotes the rational numbers, $\mathbb{R}$ the reals, and $\mathbb{C}$ the complex numbers. Fields should not be assumed to be the reals or complex numbers unless specifically stated. You should justify all assertions to receive full credit. Demonstrating an in-depth understanding of the material is the most important factor.

1.

a) Find, up to isomorphism, all abelian groups of order 72 which do not contain an element of order 9. Clearly state any results you use.

b) Recall that an automorphsim of a group G is a 1-1 and onto homomorphism of G to itself. The set of all automorphsims of G forms a group $\mathrm{Aut}(G)$. Let C be a cyclic group of order 21. Find (with proof) the order of $\mathrm{Aut}(C)$.

2. Let $f(T) = T^3 + \omega T + \sqrt{3}$, where ω is a primitive cube root of unity, let α be a root of f in $\mathbb{C}$, and set $F = \mathbb{Q}(\alpha)$. Prove that

$$4 \leq [F : \mathbb{Q}] \leq 12.$$

3. Let $R = k[X, Y]$, where k is a field. Define a ring homomorphism $f : R \to k[t]$ by $f(p(X, Y)) = p(t^2, t^3)$.

a) Prove that the kernel of f is a prime ideal P.

b) Prove that P is a principal ideal generated by $X^3 - Y^2$.

4.

a) Let $v = (1, 1, 1, 1)$ and $w = (5, 3, 3, 3)$, and let V be the subspace in $\mathbb{R}^4$ spanned by v and w. Find an orthonormal basis for V.

b) Give an example of matrices, none of which are the identity, which are 2 by 2 matrices over $\mathbb{C}$, having each of the following properties: normal, orthogonal, and self-adjoint.

5. Let

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix}.$$

and

$$B = \begin{pmatrix} 0 & 1 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Prove that A and B are similar.

6. Let T be a linear operator on a finite dimensional vector space V over $\mathbb{C}$ whose characteristic polynomial is $t^3(t-2)^2$. Find all possible Jordan normal forms for such an operator. What is the determinant of such an operator?

Algebra Qualifying Examination : January 2007

You must show all work to receive full credit. Throughout $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote respectively the integers, rational numbers, real numbers and complex numbers.

1. Consider the real valued functions $f(x) := \frac{1}{x}$ and $g(x) := \frac{x-1}{x}$ defined on $\mathbb{R}\backslash\{0,1\}$. Let G be the group generated by $f(x)$ and $g(x)$ whose group law is composition of functions. Prove that G is isomorphic to the symmetric group S_3. (16 pts)

2. Let $f(x) := x^3 - 3x - 3$ belong to the polynomial ring $\mathbb{Z}[x]$. For a prime number p, let $f_p(x)$ denote the polynomial with coefficients in $\mathbb{Z}/p\mathbb{Z}$ obtained by reducing the coefficients of $f(x)$ mod p. For what p does $f_p(x)$ have multiple roots ? (16 pts)

3. Show that the polynomial rings $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$ are not isomorphic as rings. (16 pts)

4. Let A be an $n \times n$ matrix with entries in the field F. Prove that the following numbers are the same : (a) the *row rank* of A, (b) the *column rank* of A and (c) the *determinantal rank* of A. Recall that the determinantal rank of A is the largest non-negative integer t such that determinant$(A') \neq 0$, for some submatrix A' of A obtained by deleting $n-t$ rows and columns of A. (16 pts)

5. Let $A := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and let $T : \mathbb{C}^4 \to \mathbb{C}^4$ be the linear transformation whose matrix with respect to the standard basis of $\mathbb{C}^4$ is A. Equip $\mathbb{C}^4$ with the standard inner product.

(a) Find the Jordan canonical form of A. (6 pts)

(b) Find an invertible 4×4 matrix P such that $P^{-1}AP$ is the matrix found in part (a). (6 pts)

(c) Does there exist a cyclic vector for T ? If so, find one; if not, explain why not. (6 pts)

(d) Find a two-dimensional T-invariant subspace $W \subseteq \mathbb{C}^4$. (6 pts)

(e) Find an orthonormal basis for W. (6 pts)

(f) Find the adjoint of $T|_W$. (6 pts)

Qualifying Exam **Algebra** August, 2006

Each question is given equal weight in the grading. $\mathbb{Q}$ denotes the rational numbers and $\mathbb{C}$ denotes the complex numbers. Fields should not be assumed to be the reals or complex numbers unless specifically stated. You should justify all assertions to receive full credit. Demonstrating an in-depth understanding of the material is the most important factor.

1. Let $G = S_n$ be the symmetric group on n elements, and let $\sigma = (123...n)$ be an n-cycle. Let K be the cyclic subgroup generated by σ. Prove that the order of the normalizer of K, i.e., the order of the subgroup $H = \{x \in S_n |\, x^{-1}\sigma x \in K\}$, is exactly $n \cdot \phi(n)$, where $\phi(n)$ is the Euler ϕ-function. (Recall that $\phi(n)$ is the number of positive integers less than n and relatively prime to n.)

2. Let $\alpha \in \mathbb{C}$ be a primitive 9th root of unity. Find the degree of the minimal polynomial of $\beta = \alpha + \alpha^{-1}$ over $\mathbb{Q}$ (with proof).

3. Let k be the finite field with 3 elements. Let $f(x)$ be an irreducible polynomial of degree n in $k[x]$. Prove that $k[x]/(f(x))$ is a field with 3^n elements. Prove that there are infinitely many non-associate irreducible polynomials in $k[x]$. (Two polynomials f, g are *associates* if $(f) = (g)$.)

4. Let V be a finite-dimensional inner product space over the complex numbers. Recall a linear operator T is said to be *self-adjoint* if $T = T^*$, where T^* is the adjoint of T. If T is self-adjoint, prove that $I + iT$ is invertible. (Where $i = \sqrt{-1}$.)

5. Let A be a complex 6 by 6 matrix which has exactly two eigenvalues, 0 and 1, and has rank 4. Give all possible minimal polynomials of such a matrix, and give an example of a matrix with each possible minimal polynomial.

6. Let T be a linear operator on a finite dimensional vector space V over a field F. Prove that

$$\mathrm{rank}(T^3) + \mathrm{rank}(T) \geq 2 \cdot \mathrm{rank}(T^2).$$

Algebra Qualifying Examination

January 9, 2006

In what follows $\mathbb{F}$ will always denote a field.

1. Let G be a finite group. Give a complete proof (without quoting any theorem) that there is an integer n such that $a^n = e$ for all $a \in G$.

2. Let $K \subseteq N \subseteq G$ be groups and subgroups. Assume that K is normal in N, N is normal in G and K is normal in G. Prove that there is an isomorphism
$$\frac{G/K}{N/K} \xrightarrow{\sim} G/N.$$

3. Let $R = C([0,1])$ be the ring of all continuous real valued functions on the interval $[0,1]$. Let $0 < p < 1$ and
$$I_p = \{f \in R : f(p) = 0\}.$$
Prove that I_p is a maximal ideal in R.

4. Let $\mathbb{F}$ be a field and $R = \mathbb{F}[X]$ be the polynomial ring over $\mathbb{F}$. Prove that R is a PID.

5. Let R be a commutative ring and I, J be two ideals such that $I + J = R$. Prove that the map
$$f : R/(I \cap J) \to R/I \oplus R/J \quad \textit{given by} \quad f(x + I \cap J) = (x + I, x + J)$$
is an isomorphism.

6. Let $\mathbb{F}$ be a field and $V = \mathbb{F}^n$. Think of the elements of V as row vectors. Let $v_1, v_2, \ldots, v_n \in V$ and let

$$A = \begin{pmatrix} v_1 \\ v_2 \\ \cdots \\ v_n \end{pmatrix}$$

Prove that $v_1, v_2, \ldots, v_n$ forms a basis if and only if A invertible.

7. Let V, W be two finite dimensional vector spaces over a field $\mathbb{F}$ and let $T : V \longrightarrow W$ be a linear transformation. Let $\mathcal{N}$ be the null space of T and $\mathcal{R} = T(V)$ be the range of T. Prove that

$$\dim(\mathcal{N}) + \dim(\mathcal{R}) = \dim(V).$$

8. Suppose V is a finite dimensional vector space over a field $\mathbb{F}$. Suppose $T : V \longrightarrow V$ is a linear operator on V. Suppose $c_1, c_2 \in \mathbb{F}$ be two distinct eigen values of T. Let

$$N(c_i) = \{v \in V : T(v) = c_i v\}$$

be the eigen spaces of c_i. Prove that $N(c_1) \cap N(c_2) = \{0\}$.

9. Suppose V is a vector space over $\mathbb{F}$ with $\dim V = n$. Suppose $T \in L(V, V)$ is an operator on V. Suppose W_0 is an $T-$invariant subspace of V and $v \in V \setminus W_0$. Write $W_1 = W_0 + \mathbb{F}[T]v$ and

$$I = \{f \in \mathbb{F}[X] : f(T)v \in W_0\}.$$

Let f be the minimal monic polynomial of I. Prove that $\dim W_1 = \dim W_0 + degree(f)$.

10. Let V be an inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. For $x, y \in V$, prove the Cauchy-Schwartz inequality that

$$|\,(x, y)\,| \quad \leq \quad \|\, x \,\|\|\, y \,\|$$

and that the equality holds if an only if

$$y = \frac{(y, x)}{\|\, x \,\|^2} x.$$

ALGEBRA QUALIFYING EXAMINATION : AUGUST 2005

You must justify your answers to receive full credit.

1. Recall that A_4 denotes the alternating group on 4 letters.

(a) What are the possible indices of subgroups of A_4 ? Explain. (5 points)

(b) For each response in (a), either give an example of a subgroup with that index or explain why no such subgroup exists. (15 points)

2. Let $f(X)$ be an irreducible polynomial of degree 35 with coefficients in $\mathbb{Q}$. Let $\alpha \in \mathbb{C}$ be a root of $f(X)$.

(a) Show that $\alpha^{29} \notin \mathbb{Q}$. (5 points)

(b) Show that $\mathbb{Q}(\alpha^4) = \mathbb{Q}(\alpha)$. (10 points)

3. Let R be a commutative ring and $R[X]$ the polynomial ring with coefficients in R. For an ideal $J \subseteq R$, write $J[X]$ for the set of polynomials with coefficients in J.

(a) If J is a prime ideal prove that $J[X]$ is a prime ideal. (10 points)

(b) Prove or disprove a similar statement for maximal ideals. (10 points)

4. Let S be the subspace of $\mathbb{R}^4$ spanned by the vectors (1,0,0,1) and (0,1,-1,1). Let T be the orthogonal complement of S in $\mathbb{R}^4$. Find the projection of the vector $(1, 1, 1, 1)$ onto : (a) S (b) T. (15 points)

5. Let $A := \begin{pmatrix} -5 & 0 & 0 \\ 1 & -7 & -2 \\ -1 & 2 & -3 \end{pmatrix}$ represent a linear operator T on $\mathbb{C}^3$. Determine whether or not T has a cyclic vector. (15 points)

6. Let A and B be 6×6 nilpotent matrices over a field F. Suppose that A and B have the same minimal polynomial and the same nullity. Prove that A and B are similar. (15 points)

ALGEBRA QUALIFYING EXAMINATION : MAY 2005

You must show all work to receive full credit. Each problem is worth 15 points.

1. Let G be a finite group such that $|G|$ is square-free. Assume that $x \cdot y = y \cdot x$ for all $x, y \in G$ such that the order of x is relatively prime to the order of y. Use Cauchy's theorem to prove that G is cyclic.

2. Let α denote the complex number $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot i$. Find, with proof, the minimal polynomial for α over $\mathbb{Q}$ and use it to write α^{105} as a polynomial in α of least degree.

3. Fix a prime number p and let R denote the subring of $\mathbb{Q}$ consisting of those fractions whose denominator is not divisible by p.

(a) Show that every proper ideal of R has the form $p^k R$, for some $k \geq 1$. Conclude that R is a PID with only one maximal ideal.

(b) Show that R modulo its unique maximal ideal is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

(c) Let X be an indeterminate and $R[X]$ the polynomial ring with coefficients in R. Show that the principal ideal generated by $pX - 1$ is a maximal ideal.

4. Let A be a 3×3 matrix with entries in $\mathbb{R}$. Prove that if A is not similar over $\mathbb{R}$ to a triangular matrix, then A is similar over $\mathbb{C}$ to a diagonal matrix.

5. Describe all rational canonical forms for 11×11 matrices over $\mathbb{Q}$ with minimal polynomial $(X-5)^3(X^2+5X+3)^2$.

6. Let T be a linear operator on a finite dimensional inner product space V over $\mathbb{C}$. Prove that T is self adjoint if and only if $< T(v), v >$ is a real number for every $v \in V$.

ALGEBRA QUALIFYING EXAMINATION : JANUARY 2005

You must show all work to receive full credit.

1. Let G be a finite abelian group. Prove that G is cyclic if and only if for each positive integer n, the set $\{x \in G \mid x^n = e\}$ has at most n elements. (16 points)

2. Let $f(X) = X^3 + 9X + 15 \in \mathbb{Q}[X]$.

(a) Let α be a real root of $f(X)$ and let F denote the set of all expressions of the form $\lambda_0 + \lambda_1 \cdot \alpha + \lambda_2 \cdot \alpha^2$, where each $\lambda_i \in \mathbb{Q}$. Give a rigorous proof that F is a subfield of $\mathbb{R}$ containing $\mathbb{Q}$. (10 points)

(b) Find the multiplicative inverse of α^2 in F. (8 points)

3. (a) Give an example of a commutative integral domain containing a non-zero prime ideal that is not a maximal ideal. (8 points)

(b) Let R be a principal ideal domain and $J \subsetneq R$ an ideal. Prove that the following are equivalent (10 points) :

(1) $J = aR$, for $a \in R$ an irreducible element

(2) $J = aR$, for $a \in R$, a prime element

(3) J is a prime ideal

(4) J is a maximal ideal

Recall that a non-zero element a in an integral domain is prime if $a|c \cdot d$, then $x|c$ or $x|d$ and a is an irreducible element if $a = c \cdot d$ implies c or d is a unit.

4. Let V be a finite-dimensional vector space over the field F and T be a linear operator on V. Let c be a scalar and suppose there exists a nonzero vector v in V such that $Tv = cv$. Prove that there exists a nonzero linear functional f on V such that $f \circ T = cf$. (16 points)

5. Prove the following or give a counter-example. If E_1 and E_2 are linear projections on a finite-dimensional vector space V such that $range(E_1) = range(E_2)$, then $ker(E_1) = ker(E_2)$. (16 points)

6. Let V be the space of complex $n \times n$ matrices with the inner product $(A|B) := tr(AB*)$. For each M in V, let T_M be the linear operator defined by $T_M(A) = MA$. Show that T_M is unitary if and only if M is unitary. (16 points)

Algebra Qualifying Examination

1. a) Show that a subgroup of a cyclic group is cyclic.

   b) Is the following "converse" of the above statement true: If a group **G** is such that every proper subgroup is cyclic, then **G** is cyclic.

2. a) Show that any group homomorphism $\Phi : G \to G'$, where the order of **G** is prime must either be the trivial homomorphism or a one-to-one map.

   b) Does there exist a surjective homomorphism from a group of order 14 to a group of order 6? Give reasons.

   c) How many group homomorphisms are there of Z *onto* Z?

3. a) Show that every finite integral domain is a field. Deduce that if **R** is a finite commutative ring with identity, then every prime ideal is maximal.

   b) Show that $X^3 + 3X^2 - 8$ is irreducible over the field of rational numbers Q.

   c) Is $\dfrac{Q\,[x]}{(x^2 - 5x + 6)}$ a field? Why?

4. a) Show that the field of real numbers R is not a finite field extension of Q.

   b) Prove that the algebraic closure of Q in C is not a finite extension of Q.

   (Hint: use Eisenstein criterion to show that Q has finite extensions in C of arbitrarily large degrees.)

5. Let V and W be vector spaces over the field F and let T be a linear transformation from V into W. Suppose that V is finite dimensional. Prove: rank (T) + nullity (T) = dim V.

6. Let D be the differentiation operator on the space of polynomials in F[t] of degree less than or equal to n.

   a) Find the matrix of T in the ordered basis $\{1, t, t^2, t^3, \ldots, t^n\}$.

   b) Find the Jordan form of the matrix in part (a). Justify your answer.

7. Let A be an $n \times n$ nilpotent matrix. Prove $A^n = 0$.

1. (17 pts) There are five groups listed below. Tell which ones are isomorphic to which of the others, and justify your answer. If two are not isomorphic, explain why not. C_n is the cyclic group of order n, and S_n is the symmetric group on n elements.

a) $C_2 \times C_{18}$
b) $C_6 \times S_3$
c) $S_3 \times S_3$
d) $C_2 \times C_2 \times C_9$
e) $C_6 \times C_6$

2. (6 pts each) The following three questions concern the ring $R = \mathbb{Q}[X,Y]/I$, where I is the ideal generated by the square of the ideal (X,Y), i.e., $I = (X,Y)^2$.

a) Describe (with proof) all the ideals of R.

b) Describe the units of R, that is the elements which have multiplicative inverses.

c) Prove there is a nontrivial ring homomorphism from R to the complex numbers.

3. (15 pts) Let $\mathbb{Q}$ be the field of rational numbers, and let $R = \mathbb{Q}[X]$ be a polynomial ring over $\mathbb{Q}$. Let $f(X) = X^n + a_{n-1}X^{n-1} + \ldots + a_0 \in R$. Let A be the companion matrix of f, namely

$$A = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & -a_0 \\ 1 & 0 & 0 & \ldots & 0 & -a_1 \\ 0 & 1 & 0 & \ldots & 0 & -a_2 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \\ 0 & 0 & 0 & \ldots & 1 & -a_{n-1} \end{pmatrix}.$$

Prove that $f(X)$ is both the minimal polynomial and the characteristic polynomial of A.

4. (15 pts) Find a 6 by 6 matrix A over the complex numbers such that $A^4 = 0$ but $A^3 \neq 0$. Let B be another 6 by 6 matrix satisfying these two conditions. Must A be similar to B (explain your answer)?

5. (15 pts) Let α be a root in the complex numbers of the monic polynomial $f(x) = x^3+x+1$. Set $K = \mathbb{Q}(\alpha)$. Prove or disprove the following statement: $\sqrt{-1} \in K$.

6. (10 pts each)

a) Let V be an inner product space. Define what it means to say an operator U on V is normal.

b) Let T be an operator on the inner product space V. Prove that T preserves the inner product if and only if for every $v \in V$, $||Tv|| = ||v||$.

ALGEBRA QUALIFYING EXAMINATION : MAY 2004

YOU MUST JUSTIFY YOUR ANSWERS TO RECEIVE FULL CREDIT.

1. State the class equation. Verify the class equation for S_4 listing its elements and grouping them according to the class equation .

2. A commutative integral domain V is said to be a *valuation domain* if for every non-zero $a, b \in V$, either $a|b$ or $b|a$.

(a) Show that the set of non-units form an ideal, M.

(b) Show that M is the unique maximal ideal of V.

(c) Let p be a prime number and V be the set of rational numbers $\frac{a}{b}$ in lowest terms such that p does not divide b. Show that V is a valuation domain.

3. Let $f(X) = X^3 + 2X^2 + X + 3$ be a polynomial with coefficients in the finite field $\mathbb{Z}_5$. Let α be a root of $f(X)$ and let F be the field $\mathbb{Z}_5(\alpha)$.

(a) Show $f(X)$ is irreducible over $\mathbb{Z}_5$.

(b) Describe the field F and determine the number of elements in F.

(c) Let $a, b \in F$, and let $g(X) = X^2 + aX + b$ be an irreducible polynomial in $F[X]$. Let γ be a root of $g(X)$, and set $K = F(\gamma)$. How many elements does K have?

4. (a) Let A and B be similar $n \times n$ matrices over the field F. How do $tr(A)$ and $det(A)$ compare with $tr(B)$ and $det(B)$? Justify your answer.

(b) Consider the matrices

$$A = \begin{pmatrix} 1 & 1 & -4 & 9 \\ 0 & 1 & -4 & 10 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & -4 & 10 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Are they similar over $\mathbb{C}$? Over $\mathbb{R}$?

5. Let V be the vector space of real polynomials of the form

$$f(X) = a_8X^8 + a_7X^7 + a_6X^6 + a_5X^5$$

and W the subspace of polynomials $f \in V$ satisfying $a_8 + \cdots + a_5 = 0$.

(a) Calculate $dim(W)$.

(b) Let $<,>$ be the inner product on W given by $< f, g >= \int_0^1 f \cdot g \, dx$, for all $f, g \in W$. Find an orthonormal basis for W.

6. Let V be an n-dimensional vector space over $\mathbb{R}$ and $T : V \to V$ a linear operator. Show that T has a non-zero eigenvector in each of the following cases : (a) n is odd (b) $n > 0$ and T is self-adjoint.

Algebra Qualifying Examination : January 2004

Throughout, **Q**, **R** and **C** denote, respectively, the fields of rational, real and complex numbers and **Z** denotes the ring of integers.

1. Let P be a Sylow 2-subgroup of S_5, the symmetric group on 5 elements.

   a) What is the order of P ?
   b) What are the possible orders of the elements of P, and how many elements have each given order ?

2. Let **C*** be the multiplicative group of non-zero elements in **C**.

   a) Let x in **C*** be an element of finite order. Show that x is algebraic over **Q**.
   b) Assume n divides m. Show that $\mathbf{Z}_n \times \mathbf{Z}_m$ cannot be isomorphic to a subgroup of **C***.
   c) For each positive integer n, find a cyclic subgroup of **C*** of order n.

3. Let A be the ring of continuos real valued functions on [0,1] (with point-wise addition and multiplication). For t in [0,1], set $M_t = \{ f \text{ in } A \mid f(t) = 0\}$. Prove that M_t is a maximal ideal (including details that M_t is an ideal).

4. Let F be a field. For a positive integer n, let I_n denote the nxn identity matrix over F.

   a) If F = **R**, prove that there do not exist nxn matrices A, B over F satisfying $AB-BA = I_n$.
   b) If F = $\mathbf{Z}_2$, show that there exist 2x2 matrices A,B over F satisfying $AB-BA = I_2$.

5. Let V denote the vector space of 2x2 matrices over **C** and set B = $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Define a linear operator T : V → V by T(A) = AB - BA, for all A in V. Find the Jordan canonical form of T.

6. Let V be a finite dimensional inner product space over **C** and T a linear operator on V.

   a) Show that the adjoint of T is unique.
   b) Now assume that V is the vector space of nxn complex matrices with inner product given by <A,B> : = tr(B*A), where B* denotes conjugate transpose. Fix an nxn matrix L and let T : V → V be the linear operator on V given by T(A) = LA, for all A in V. Show that T* is left multiplication by L*, i.e., T*(A) = L*A, for all A in V.

ALGEBRA QUALIFYING EXAMINATION : AUGUST 2003

1. Let G be a finite group and recall that elements x and y in G are said to be *conjugate* if there exists $g \in G$ such that $y = g^{-1}xg$.

(a) Show that if G has only two conjugacy classes, then $|G| = 2$. (7 pts)

(b) Find the number of conjugacy classes in the symmetric group S_4. (8 pts)

2. (a) Let $f(x)$ be a degree four monic polynomial with integer coefficients. Answer the following questions true or false (no justification required) :

(i) If $f(x)$ factors non-trivially over $\mathbb{Q}$, then it factors non-trivially over $\mathbb{Z}$. (3 pts)

(ii) $f(x)$ factors as a product of quadratic polynomials over $\mathbb{R}$. (3 pts)

(iii) $f(x)$ factors as a product of linear polynomials over $\mathbb{C}$. (3 pts)

(b) Prove that $x^4 + x + 1$ is irreducible over $\mathbb{Q}$. (11 pts)

3. Let $R := \mathbb{Z}[x]$ be the ring of polynomials with integer coefficients.

(a) Describe the ideal $5R$ and show that it is a prime ideal. (5 pts)

(b) Let $M := (5, x^2 + 2)R$. Show that M is a maximal ideal. (5 pts)

(c) How many elements are in the field R/M ? (5 pts)

4. Find a 3×3 matrix A over $\mathbb{R}$ such that $A \neq I_3$, yet $A^3 = I_3$. (15 pts)

5. Fix a 5×5 matrix A over $\mathbb{C}$. Define I_A to be the ideal of polynomials $f(x)$ in $\mathbb{C}[x]$ such that $f(A) = 0$. Suppose I_A is generated by $(x-1)^3(x-2)^2$ and $(x-1)^2(x-2)^3$. What are the possible Jordan cannonical forms for A ? (Warning : the given generating set for I_A is not necessarily a minimal generating set.) (15 pts)

6. Consider the complex vector space $\mathbb{C}^3$ with the usual inner product. For each $v, w \in \mathbb{C}^3$, let $T_{v,w}$ be the map on $\mathbb{C}^3$ defined by $T_{v,w}(x) = < x, w > v$. Show that :

(a) $T_{v,w}$ is a linear operator on $\mathbb{C}^3$. (5 pts)

(b) $(T_{v,w})^* = T_{w,v}$. (5 pts)

(c) $trace(T_{v,w}) = < v, w >$. (5 pts)

(d) When is $T_{v,w}$ self-adjoint ? (5 pts)

Qualifying Exam Algebra May, 2003

1. Let G be a finite group. The *commutator subgroup* of G, denoted $[G, G]$, is the set of all finite products of elements of the form $g^{-1}h^{-1}gh$, where g and h range over all elements of G.

a) (8 pts) Prove that $[G, G]$ is a subgroup of G, and prove that it is a normal subgroup.

b) (7 pts) Let H be a normal subgroup of G. Prove that G/H is abelian if and only if $[G, G] \subseteq H$.

2. (10 pts each) The following two questions concern the ring $R = \mathbb{Z}[X]/I$, where I is the ideal generated by the elements $X^2 + 1$ and 6.

a) Describe (with proof) all the maximal ideals of R.

b) Prove that R is a finite ring, and find the number of elements of R.

3. (15 pts) Let $\mathbb{Q}$ be the field of rational numbers, and let $R = \mathbb{Q}[X]$ be a polynomial ring over $\mathbb{Q}$. Let $f(X) \in R$ be a monic irreducible polynomial in R of degree n, and let α be a complex root of $f(X)$. Set $K = \{a_0 + a_1\alpha + \ldots + a_k\alpha^k + \ldots + a_{n-1}\alpha^{n-1} \mid a_0, \ldots, a_{n-1} \in \mathbb{Q}\}$. Prove that K is a subfield of the complex numbers.

4. (15 pts) Let A be a 4×4 matrix with entries in the complex numbers. Assume that $A^3 - 3A^2 + 2A = 0$. Using Jordan normal form, classify up to similarity all possible such A.

5. (15 pts) Let V be an finite dimensional vector space over a field F, and let T be a linear operator on V. Prove that $V = \mathrm{Ker}(T) \oplus \mathrm{Image}(T)$ if and only if $\mathrm{Ker}(T) = \mathrm{Ker}(T^2)$. Recall that $\mathrm{Ker}(T) = \{v \in V \mid T(v) = 0\}$ and $\mathrm{Image}(T) = \{T(v) \mid v \in V\}$.

6. (10 pts each) Let V and W be finite dimensional inner product spaces over the complex numbers.

a) Suppose that $T : V \to W$ is a linear operator and let T^* be its adjoint, i,e., $T^* : W \to V$ satisfies $< T(v), w >=< v, T^*(w) >$ for all $v \in V$ and $w \in W$. Prove that T is injective if and only if T^* is surjective.

b) Suppose that $T : V \to V$ is a linear operator on V which is self-adjoint. Prove that $T^3 = I$ implies that $T = I$, where I is the identity operator.

Algebra Qualifying Examination August '02.

1. Determine whether the polynomial in $\mathbb{Z}[x]$ satisfies the Eisenstein Criteria for irreducibility over Q

$$2x^2 - 25x^3 + 10x^2 - 30$$

2. Show that a group with at least two elements but with no proper nontrivial subgroups must be finite and of prime order.

3. Let G be a group of order pq, p, q prime numbers. Show that every proper subgroup of G is cyclic.

4. Show that 1 and $p - 1$ are the only elements of field $\mathbb{Z}_p$ that are their own multiplicative inverse.

5. Let E be an extension field of a finite field F, where F has q elements. Let $\alpha \in E$ be algebraic over F of degree n. Prove that $F(\alpha)$ has q^n elements.

6. a) Show that there exists an irreducible polynomial of degree 3 in $\mathbb{Z}_3[x]$.

   b) Use (5) and (a) to show that there exists a field of 27 elements.

   c) Does there exist a field with 28 elements? Explain.

7. Let $A = \begin{bmatrix} 2 & -1 & -3 & 0 & 0 \\ 1 & 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & -3 \end{bmatrix}$

   a) Find the characteristic polynomial of A.
   b) Find the minimal polynomial of A.
   c) Find the Jordan form of A.

8. Let n be a positive integer. Let V be the real space of all polynomials over $\mathbb{R}$ of degree at most n. Let $T : V \to V$ be the linear transformation defined by $T(f(t)) = tf'(t) - (n + 1)f(t)$, where $f'(t)$ represents the derivative of $f(t)$. Show that $\det(T) \neq 0$.

9. Let W be a subspace of a finite dimensional vector space V. Prove: $\dim W + \dim W^0 = \dim V$.

Qualifying Exam Algebra January, 2002

1. (Group Theory) (10 pts each)

a) Find the maximal order of an element in the symmetric group S_{10}. Explain what results you use to find the maximal order.

b) Let G be a group. Prove that a subgroup H is normal in G if and only if H is the kernel of a homomorphism from G to another group.

2. (Rings) (10 pts) Recall that the characteristic of a commutative ring R with identity 1 is the least integer n such that $n \cdot 1 = 0$. Find (with proof) the characteristic of the ring $R = \mathbb{Z}_6 \times \mathbb{Z}_2 \times \mathbb{Z}_{15}$.

3. (Integers, Fields and Polynomials) (10 pts each)

a) Prove that $x^3 + 3x^2 - 8$ is irreducible over the rational numbers.

b) Let $K \subseteq L$ be fields. Set $F = \{\alpha \in L | \, \alpha \text{ is algebraic over } K\}$. Prove that F is a field.

c) Prove that for all integers n, $n^{37} - n$ is divisible by 114.

4. (Matrices) (15 pts) Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 4 \end{pmatrix}.$$

Find the characteristic polynomial, the minimal polynomial, and the Jordan normal form of A.

5. (Linear Operators and Vector Spaces)

a) (8 pts) Let V be an n-dimensional vector space over a field F, with $n \geq 2$. Suppose that N is a linear operator on V such that $N^{n-1} \neq 0$ but $N^n = 0$. Prove that there is not an operator T on V such that $T^2 = N$.

b) (7 pts) Let V be a finite dimensional vector space, and suppose that T and S are two linear operators on V which commute. Let λ be an eigenvalue of T, and set $W = \{v \in V | \, Tv = \lambda v\}$. Prove that $S(W) \subseteq W$.

6. (Inner Product Spaces) (10 pts.) Let T be a self-adjoint linear operator on a finite dimensional complex inner product space. Let I be the identity operator. Prove that $S = I + iT$ is invertible. (Hint: consider $(Sv|Sv)$.)

Qualifying Exam | Algebra | August, 2001

There are two pages to this exam.

1. (Group Theory) Let G be a simple group of order 168 (such a group does exist). For each of the following numbers, say whether or not G has a subgroup of that order. Give reasons. 'Maybe' is not an accepted answer.

a) (4 pts) Order 4.
b) (3 pts) Order 8.
c) (3 pts) Order 11.
d) (3 pts) Order 21.
e) (3 pts) Order 42.
f) (4 pts) Order 84.

2. (Fields and Polynomials) (10 pts) Let $\alpha = \sqrt{3}$ in the complex numbers and let β be a cube root of 7. Suppose that a, b, c, d, e, f are rational numbers and

$$a + b\alpha + c\beta + d\beta^2 + e\alpha\beta + f\alpha\beta^2 = 0.$$

Prove that $a = b = c = d = e = f = 0$.

3. (Group Theory)

a) (10 pts) Let G be a group of order 105. Prove that G is not simple.
b) (10 pts) List all abelian groups (up to isomorphism) of order 225.

4. (Inner Product Spaces) Let V be a finite-dimensional inner product space over the complex numbers.

a) (5 pts) Let T be a linear operator on V. Suppose that for all $v \in V$, $(Tv|v) = 0$. Prove that the only possible characteristic value of T is 0.

b) (10 pts) Let T be a linear operator on V. Set $U = \operatorname{Ker}(T)$ and set $W = \operatorname{Im}(T^*)$. Prove that $U = W^{\perp}$.

5. (Linear Operators) (20 pts) Let

$$A = \begin{pmatrix} 3 & -4 & -4 \\ -1 & 3 & 2 \\ 2 & -4 & -3 \end{pmatrix}.$$

Find the characteristic polynomial, the minimal polynomial, the Jordan normal form and the rational canonical form of A.

6. (Vector Spaces and Linear Functionals)

a) (7 pts) Let V be the vector space of all n by n matrices over a field F. Fix an n by n matrix B, and define a linear operator $T : V \to V$ by $T(A) = AB - BA$. Prove that $\det(T) = 0$.

b) (8 pts) Let V be a finite dimensional vector space over a field F. Prove that V is isomorphic to its double dual V^{**}.

Qualifying Exam | Algebra | May, 2001

1. (Group Theory) Give examples of each of the following:

a) (2 pts) A nonabelian finite group with a normal subgroup of index 2.

b) (3 pts) Two finite nonabelian, non-isomorphic groups of the same order.

c) (2 pts) A finite nonabelian simple group.

d) (3 pts) A finite abelian group which is not isomorphic to the product of two cyclic groups.

2. (Fields and Polynomials)

a) (10 pts) Let α be a nonzero complex number. Suppose that α is the root of a monic polynomial of degree n with rational coefficients. Prove that $\beta = \alpha^2 + \alpha^{-1}$ is also the root of a monic polynomial with rational coefficients of degree at most n.

b) (10 pts) Prove that $x^6 + x^3 + 1$ is irreducible over the rational numbers.

3. (Group Theory)

a) (5 pts) State the Class Equation for a finite group G.

b) (5 pts) Let G be a group of order 81, and suppose that the center of G has order 9. How many conjugacy classes does G have? (Justify your answer.)

c) (10 pts) Let G be a group of order $2^4 \cdot 3 \cdot 7^2$. Prove that G is not simple.

4. (Inner Product Spaces) Let V be a finite dimensional inner product space over the complex numbers.

a) (4 pts) If W is a subspace of V, prove that $W^{\perp} \oplus W = V$.

b) (3 pts) Let $T : V \to V$ be a linear operator. If $(Tv|w) = 0$ for all $v, w \in V$, prove that T is the zero operator.

b) (8 pts) Let $T : V \to V$ be a self-adjoint linear operator. Prove that all the characteristic values of T are real, and prove that characteristic vectors associated to distinct characteristic values are orthogonal.

5. (Linear Operators)

a) (10 pts) Let

$$A = \begin{pmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Find the rational normal form and the Jordan canonical form for A.

b) (10 pts) Classify up to similarity all 4 by 4 complex matrices A which satisfy $A^3 = A^2$.

6. (15 pts) (Vector Spaces and Linear Functionals) Let V be a finite dimensional vector space over a field F. Assume $\dim(V) = n$. Let $f_1, \ldots, f_n \in V^*$. Define a linear operator

$$T : V \to F^n$$

by $T(v) = (f_1(v), \ldots, f_n(v))$ for $v \in V$. Prove that T is an isomorphism if and only if $f_1, \ldots, f_n$ are linearly independent.

Qualifying Exam **Algebra** January, 2001

1. If G is a group and H is a subgroup of G, recall that $N_G(H) = \{x \in G \,|\, x^{-1}Hx \subseteq H\}$.

a) (5 pts) State the three theorems of Sylow.

b) (5 pts) Let G be a finite group, and let p be a prime number dividing the order of G. Let P be a p-Sylow subgroup of G. Prove that P is the unique p-Sylow subgroup of $N_G(P)$.

c) (10 pts) Let G be a finite group, and let p be a prime number dividing the order of G. Let P be a p-Sylow subgroup of G. Prove that $N_G(N_G(P)) = N_G(P)$.

2.

a) (5 pts) State Gauss's Lemma.

b) (10 pts) Let α be a complex number. Assume that there is a monic polynomial $f(X)$ with integer coefficients such that $f(\alpha) = 0$. Prove that the minimal polynomial of α over the rational numbers also has integer coefficients. (Recall that by definition, the minimal polynomial is monic.)

3. Let α be a primitive 8th root of unity in the complex numbers.

a) (5 pts) Find the minimal polynomial for α over the rational numbers (with proof).

b) (5 pts) Write the inverse of $\alpha + 2$ as a polynomial in α with coefficients in the rational numbers.

c) (5 pts) Let $R = \mathbb{Z}[\alpha] = \{f(\alpha) \,|\, f(X) \in \mathbb{Z}[X]\}$. What is the cardinality of $R/(5R)$?

d) (5 pts) Describe all the maximal ideals in the ring $R/(5R)$.

4. (15 pts) Let V be a finite dimensional vector space over a field F, and let $T : V \to V$ be a linear transformation. Write $p(X)$ for the minimal polynomial of T and $q(X)$ for the characteristic polynomial of T. Prove that for all $\lambda \in F$, $q(\lambda) = 0$ if and only if $p(\lambda) = 0$.

5. (15 pts) Let V be the subspace of $\mathbb{R}^4$ generated by the vectors $[1, 0, 1, 0]$, $[1, 0, 1, 1]$, and $[-1, 0, 1, 0]$. Find an orthonormal basis for V.

6. Let A be a 6 by 6 nilpotent matrix over the complex numbers.

a) (5 pts) Prove that $A^6 = 0$.

b) (10 pts) Assume that $A^3 = 0$ and $A^2 \neq 0$. Find all possible Jordan canonical forms for such a matrix.

ALGEBRA QUALIFYING EXAM : JANUARY 2000

1. Let G be a group and $N_1, \dots, N_k$ subgroups of G.

(a) What conditions must be met in order for G to be the internal direct product of $N_1, \dots, N_k$?

(b) If G is the internal direct product of $N_1, \dots, N_k$, show $G \cong N_1 \times \cdots \times N_k$.

2. Let $(G, +)$ be a finite abelian group and n a positive integer. Let $\phi : G \to G$ be the abelian group homomorphism defined by $\phi(g) = ng$, for all $g \in G$. Here, we are writing ng for $g + \cdots + g$, n times. Prove that $ker(\phi) \cong G/im(\phi)$.

3. Let R be a unique factorization domain with quotient field K. If $f(x) \in R[x]$ is a primitive polynomial, show that $f(x) \cdot K[x] \cap R[x] = f(x) \cdot R[x]$.

4. Let $\epsilon := e^{\frac{2\pi i}{6}}$. (a) Show that ϵ satisfies $x^2 - x + 1 = 0$.

(b) Verify that $[\mathbb{Q}(\epsilon) : \mathbb{Q}] = 2$.

(c) Show that $\phi : \mathbb{Q}(\epsilon) \to \mathbb{Q}(\epsilon)$ defined by $\phi(a + b\epsilon) = (a + b) - b\epsilon$ is an automorphism. (Note, $a, b \in \mathbb{Q}$.)

(d) Explain why id and ϕ are the only automorphisms of $\mathbb{Q}(\epsilon)$.

5. Let V be the vector space of complex polynomials having degree five or less and $D : V \to V$ the linear transformation given by differentiation. Find the Jordan canonical form for $D^2 := D \circ D$.

6. Let V be an n-dimensional vector space over the field K and $T : V \to V$ a linear transformation.

(a) Define the characteristic polynomial $f_T(x)$ of T. Why is your answer well-defined ?

For parts (b) and (c) assume there exists $v \in V$ such that $\{v, T(v), \dots, T^{n-1}(v)\}$ is a basis for V.

(b) Show that the matrix of T with respect to the given basis equals the companion matrix of $f_T(x)$.

(c) Use the given basis to show that the conclusion of the Cayley-Hamilton theorem holds for T, i.e., $f_T(T) = 0$.

(e) How can the foregoing ideas be used to give a proof of the Cayley-Hamilton theorem in general ?

ALGEBRA QUALIFYING EXAMINATION : AUGUST 2000

1. For $n \geq 1$, let D_n denote the dihedral group (i.e., the group of symmetries of a regular n-sided polygon).

(a) Define D_n in terms of generators and relations.

(b) Show that D_4 is not isomorphic to Q_8, the quaternion group.

(c) Does D_9 have an element of order 4 ? Justify your answer.

2. Let G be a finite group and suppose there exists an element in G which has only two distinct conjugates. Show that G has a non-trivial normal subgroup.

3. Let F be a field and $F[X]$ denote the ring of polynomials with coefficients in F.

(a) State the division algorithm, as it applies to elements of $F[X]$.

(b) Using the division algorithm, give a procedure for finding the greatest common divisor of any two non-zero elements of $F[X]$.

(c) Let $F := \mathbb{Z}_5$, $f(X) := X^5 + X^4 + X^3 + X^2 + X + 1$ and $g(X) := X^3 + 4$. Find the GCD of $f(X)$ and $g(X)$.

(d) Find $l(X), h(X) \in \mathbb{Z}_5[X]$ such that $GCD(f,g) = l(X) \cdot f(X) + h(X) \cdot g(X)$.

4. Let V be a finite dimensional inner product space over $\mathbb{C}$ and $T : V \to V$ a normal linear transformation, i.e., $TT^* = T^*T$, where T^* denotes the Hermetian adjoint of T. Prove that if $T^k = 0$, some k, then $T = 0$. (If you prefer, you may prove the corresponding statement for square matrices over $\mathbb{C}$.)

5. Describe the 'canonical form' taken by nilpotent matrices, i.e., if A is a square nilpotent matrix over the field F, describe its rational (or Jordan) canonical form. Find the nilpotent form for the matrix

$$\begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

ALGEBRA QUALIFYING EXAMINATION : AUGUST 1999

1. Define unique factorization domain (UFD). Let R be a UFD. Assume that α belongs to the quotient field of R and satisfies a monic polynomial with coefficients in R. Show that $\alpha \in R$.

2. Fix an integer $n > 1$ and let $X \subseteq \mathbb{C}$ denote the roots of the polynomial $x^n - 1$.

(a) Show that X is a cyclic group of order n under complex multiplication.

(b) A generator of X is called a *primitive n^{th} root of unity.* How many primitive n^{th} roots of unity are there ?

(c) Let ϵ be a primitive n^{th} root of unity and σ, τ automorphisms of the field $\mathbb{Q}(\epsilon)$. Show that $\sigma\tau = \tau\sigma$.

(d) Let ϵ be a primitive cube root of unity. Show by explicit calculation that there exists an automorphism σ of $\mathbb{Q}(\epsilon)$ taking ϵ to ϵ^2.

3. Let G be a finite group and p a prime number dividing $|G|$. Let S denote the set of p-tuples a of elements of G such that the product of the coordinates of a equals e. That is, $S := \{(g_1, \cdots, g_p) \mid g_i \in G \text{ and } g_1 g_2 \cdots g_p = e\}$.

(a) Show that if $a \in S$, then any cyclic permutation of a belongs to S. I.e., if $(g_1, \cdots, g_p)$ belongs to S, then $(g_i, g_{i+1}, \cdots, g_p, g_1, \cdots, g_{i-1})$ belongs to S for all $1 \leq i \leq p$.

(b) For $a, b \in S$, define $a \sim b$ if b is a cyclic permutation of a. Show that $\sim$ is an equivalence relation on S.

(c) Let $[a]$ denote the equivalence class of $a \in S$. Show that $|[a]| = 1$ or $|[a]| = p$.

(d) Calculate $|S|$.

(e) Use the foregoing to show that G has an element of order p.

4. Let V be a finite dimensional vector space and $T : V \to V$ a linear transformation satisfying $rank(T) = rank(T^2)$. Show that $V = ker(T) \oplus range(T)$.

5. State either the Jordan canonical form theorem or the rational canonical form theorem. Find the Jordan canonical form AND the rational canonical form for the matrix

$$\begin{pmatrix} 0 & 3 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 10 & 0 & -3 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$